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On boundaries of bicombable spaces

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O brzegach przestrzeni biczesalnych

praca doktorska
napisana pod opieką

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Abstract

In the first part of the thesis we initiate systematic study of EZ-structures (and associated boundaries) of groups acting on spaces that admit consistent and conical (equivalently, consistent and convex) geodesic bicombings. Such spaces recently drew a lot of attention due to the fact that many classical groups act 'nicely' on them. We rigorously construct EZ-structures, discuss their uniqueness (up to homeomorphism), provide examples, and prove some boundary-related features analogous to the ones exhibited by CAT(0) spaces and groups, which form a subclass of the discussed class of spaces and groups.

In the second part of the thesis we give complete characterizations (in terms of nerves) of those word hyperbolic Coxeter groups whose boundary is homeomorphic to the Sierpiński curve and to the Menger curve, respectively. The justification is mostly an appropriate combination of various results from the literature.

Streszczenie

W pierwszej części rozprawy kładziemy podwaliny pod ustrukturyzowane badania nad EZ-strukturami — i powiązanymi z nimi brzegami — dla grup działających na przestrzeniach metrycznych mających zgodne, stożkowe (równoważnie: zgodne, wypukłe) biuczesanie geodezyjne. Ostatnimi czasy takim przestrzeniom poświęcono sporo uwagi badawczej ze względu na fakt, że wiele klasycznie rozważanych rodzin grup działa na nich w sposób interesujący z geometrycznego punktu widzenia. Z dbałością o szczegóły konstruujemy EZ-struktury, poruszamy też temat ich jednoznaczności (rozważanej z dokładnością do homeomorfizmu) i przedstawiamy ich przykłady; następnie dowodzimy, że rozważane w tej części rozprawy klasy przestrzeni i grup wykazują pewne związane z brzegami własności analogiczne do tych przejawianych przez pewną ich podklasę — grupy i przestrzenie CAT(0).

W drugiej częsci rozprawy wskazujemy — wyrażone poprzez warunki dotyczące nerwów — pełne charakteryzacje tych hiperbolicznych grup Coxetera, których brzeg jest homeomorficzny z dywanem Sierpińskiego, i tych, których brzeg jest homeomorficzny z krzywą Mengera. Uzasadnienie jest w dużej części kombinacją różnych wyników występujących w literaturze.

Contents

Introduction to the thesis

This two-part thesis oscillates around the topic of boundaries at infinity, treating the subject from a geometric viewpoint.

Part A, titled 'On boundaries of bicombable groups', is based on the preprint of the same title by the author of this thesis, and focuses on developing the boundary-related parts of the theory of spaces that admit geodesic bicombings.

Part B, titled 'Complete characterisations of hyperbolic Coxeter groups with Sierpiński curve boundary and with Menger curve boundary', is based on the paper of the same title, joint with Michael Kapovich and Jacek Świątkowski, which has just appeared as an online-first article in Fundamenta Mathematicae [DKŚ24]. This part is dedicated to a specific boundary-related problem — we state and prove the eponymous characterisations. The individual contributions to this paper are as follows. The outline of the proof was suggested by Michael Kapovich. The author of this thesis and Jacek Świątkowski filled this outline with details. In particular, the author of this thesis found the proofs of two key technical parts of the aforementioned outline: the observation from combinatorial group theory in Subsection 1.5, and the claim in the proof of Proposition 1.9.

Acknowledgements. I would like to thank my advisors — Jan Dymara and Damian Osajda — and Jacek Świątkowski for various forms of support during the preparation of this thesis.

A On boundaries of bicombable spaces

This part of the thesis is based on the preprint sharing the title with this part written by the author of this thesis. See the introduction to the thesis for more details.

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0. Introduction

Non-positive curvature (NPC) plays a prominent role in the Geometric Group Theory, and appears there in many forms. Two of the most important instances of NPC are CAT(0) [BH99] and (Gromov/word) hyperbolic [Gro87] spaces and groups. Recently, a lot of attention has been attracted by another form of non-positive curvature, in a sense generalising these two concepts, and reaching far beyond the world of CAT(0) and hyperbolic spaces and groups. Namely, we focus our attention on geodesic metric spaces that admit so-called $\mathfrak{ccc}$ geodesic bicombings. A *geodesic bicombing* σ on a space X is a continuous function $X \times X \times [0, 1] \to X$ which is a continuous choice of geodesics in X, and a bicombing is a $\mathfrak{ccc}$ *bicombing* if it satisfies properties called consistency and conicality (or, equivalently, the ones called consistency and convexity) — see Definition 1.1 and Remark 1.2 for more details.

CAT(0) spaces are particular examples of $\mathfrak{ccc}$-bicombable spaces, and with every word hyperbolic group there is a canonically associated $\mathfrak{ccc}$-bicombable space — namely the so-called injective hull of the word hyperbolic group. Further important examples of $\mathfrak{ccc}$-bicombable spaces come from the realm of injective metric spaces [Lan13; DL15] — see Corollary II(i); and other important examples of groups acting 'nicely' on $\mathfrak{ccc}$-bicombable spaces include Helly groups [Cha+24]— see Corollary II(ii). In particular, the last class of groups includes many classical families — FC-type Artin groups, some lattices in Euclidean buildings, Garside groups, some small cancellation groups, as well as all CAT(0) cubical groups, and all word hyperbolic groups. For some of these groups the structure of a group acting geometrically on a $\mathfrak{ccc}$-bicombable space is the only known form of non-positive curvature, and this allowed to exhibit many important features of such groups and to answer a few open questions about them. The theory of Helly groups, groups acting geometrically on injective metric spaces, and, more generally, groups acting in a controlled way on $\mathfrak{ccc}$-bicombable spaces is currently being intensively developed, bringing many new achievements: [Lan13; DL15; DL16; Mie17; Bas18; Cha+20; HO21a; HO21b; GMS22; Hae22b; Hae22a; Hae23a; Hae23b; HHP23; Hod23; Cha+24; OV24].

In the current article, we initiate a systematic study of boundaries of $\mathfrak{ccc}$-bicombable spaces, in particular, such spaces acted upon geometrically by a group. More precisely, for a group G acting geometrically on a $\mathfrak{ccc}$-bicombable space X we construct and study its EZ-structures. Following [Bes96; FL05; Dra06; OP09], where Z-structures and their equivariant version, EZ-structures, have been defined in ways slightly differing in some details, an *EZ-structure* $\overline{X}$ is a G-equivariant compactification of X where the boundary $\partial X := \overline{X} \setminus X$ is a 'small' subset of $\overline{X}$ — see Definition 1.5 and the discussion below it.

The visual-boundary compactification of a CAT(0) space admitting a geometric group action, or the compactification of a suitable Rips complex of a word hyperbolic group by adding its Gromov boundary, are the two most important — and historically the first — examples of EZ-structures. Already the existence of an EZ-structure has very important consequences, e.g. it implies the Novikov Conjecture for the group in the torsion-free case. Furthermore, the topology of the boundary reflects some algebraic properties of the group (see e.g. [Bow98; Swe99; PS09]) and various topological invariants of the boundary are invariants of the group (see e.g. [BM91; Bes96; Dra06; GO07]). Besides CAT(0) and word hyperbolic groups, (E)Z-structures have been constructed for various other families of groups in e.g. [Dah03; OP09; Tir11; Mar14; Gui14; Pie18; GMT19; GMS22; EW23; Cha+24]. For the existence of a Z-structure in the torsion-free case, it is required for a group to be of type F, that is to admit a finite classifying space. It is a well-known open question whether all groups of type F possess (E)Z-structures [Bes96, Question in 3.1].

Recall that an action of a group G on a metric space X via isometries is *geometric* if it is *proper*, that is the set $\{g \in G : gK \cap K \neq \emptyset\}$ is finite for all compact sets $K \subseteq X$, and *cocompact*, that is the quotient $G \backslash X$ is compact. Our first main result is that in the case of groups acting geometrically on $\mathfrak{ccc}$-bicombable spaces an EZ-structure indeed exists.

Theorem I. *Let G be a group acting geometrically on a finite-dimensional proper geodesic metric space (X, d), which possesses a $\mathfrak{ccc}$, G-equivariant bicombing σ. Then G admits an EZ-structure.*

Although the existence of an EZ-structure — following [DL15], where the authors give the construction of the compactification and prove several of the Z-structure–properties for it — was claimed in [Cha+24], there was no rigorous proof there. We present two independent — and equivalent, see Corollary 3.7 — constructions of the EZ-structure:

one using the construction via the Gelfand dual from [EW23], and one going along the more standard approach, known from the CAT(0) and word hyperbolic cases, namely via equivalence classes of infinite rays (compatible with the lines) of the bicombing, initiated in the aforementioned article [DL15] — see Subsections 3.1 and 2.1, respectively, for more details. For a space X admitting a ccc bicombing σ, we denote below by $\partial_\sigma X$ the boundary resulting from the construction from [DL15], and put $\overline{X}_\sigma := X \cup \partial_\sigma X$.

Since [AP56], injective metric spaces have appeared and have been rediscovered in many research areas of mathematics and computer science, and are known under many different names: in the contexts closest to ours — in topology and metric geometry — they are known as *hyperconvex spaces*, or *absolute retracts* or *injective objects* (in the category of metric spaces with 1-Lipschitz maps); they have also appeared in e.g. functional analysis and fixed point theory [Sin79; Soa79], and theoretical computer science [CL94]. In this article we use the following definition, known under the name *'hyperconvexity'*: a metric space (X, d) is injective if for every family of points $x_i \in X$ and radii $r_i > 0$ such that $d(x_i, x_j) \leq r_i + r_j$ for every $i, j \in I$ the intersection $\bigcap \overline{B}(x_i, r_i)$ (of closed balls) is non-empty. As is shown in [Isb64], each metric space X admits an *injective hull* $E(X)$ — the 'smallest' injective metric space that contains X. Then the *combinatorial dimension* [Dre84] is defined as $\dim_{\mathrm{comb}} X = \sup\{\dim E(Y) : Y \subseteq X, |Y| < \infty\}$. For more details about the notion of an injective metric space and of the injective hull $E(X)$ of a metric space X one may also see e.g. [Lan13]. For more about Helly graphs and groups one may see e.g. [Cha+24].

Corollary II. *A group G acting geometrically on:*

(i) either a finite-dimensional proper injective metric space X

(ii) or a locally finite Helly graph Γ

admits an EZ-structure.

Remark III. In view of [DL15, Theorems 1.1 and 1.2], if X is a proper metric space of finite combinatorial dimension or such that every bounded subset of X has finite combinatorial dimension, and admits a conical bicombing, then X admits a ccc, reversible bicombing, which is a unique convex bicombing on X, therefore is equivariant with respect to the full isometry group of X.

Proof. (i) By [Lan13, Proposition 3.8(1)], X admits a conical bicombing and, since for each subspace $Y \subseteq X$ the injective hull $E(Y)$ embeds into $E(X) = X$ (see e.g. [Lan13, Proposition 3.5(1)]), we have that $\dim_{\mathrm{comb}} X \leq \dim X < \infty$. Therefore the claim follows by Remark III and Theorem I.

(ii) By [Cha+24, Theorem 6.3], the action of G extends to a geometric action on the injective hull $E(\Gamma)$, which is a proper injective metric space, and which is finite-dimensional as a locally finite polyhedral complex on which the group G acts cocompactly. The claim follows by (i). $\square$

Croke and Kleiner [CK00] showed that in the case of a group acting geometrically on two CAT(0) spaces X and Y, their boundaries ∂X and ∂Y may be non-homeomorphic. Since CAT(0) spaces are $\mathfrak{ccc}$-bicombable, this provides examples of non-homeomorphic boundaries of a group acting geometrically on a $\mathfrak{ccc}$-bicombable space. However, even in the CAT(0) world, restricting to some particular cases of spaces and/or groups brings one to a situation when a boundary of a group is well defined, up to homeomorphism. Such phenomenon is called 'boundary rigidity' [Rua99; Hos03; HK05]. Since injective spaces form one of the most important examples of $\mathfrak{ccc}$-bicombable spaces, and their geometry seems very 'rigid' in a sense, a natural question is whether the boundary in this case is unique up to homeomorphism. We give a negative answer in the following theorem.

Theorem IV (Theorem 4.1)**.** *There exists a group G acting geometrically on two proper finite-dimensional injective metric spaces X^1, X^2 with convex bicombings σ^1, σ^2, respectively, such that $\partial_{\sigma^1} X^1$ and $\partial_{\sigma^2} X^2$ are not homeomorphic.*

The example of the group G from the above theorem is the original Croke–Kleiner example [CK00]. The two injective spaces X^1, X^2 are basically the Croke–Kleiner polygonal complexes carefully equipped with two different injective metrics. It is intriguing that, unlike in the CAT(0) case, the injective metric structure imposes severe restrictions on the gluing pattern within the complexes. In particular, we are able to produce only two homeomorphism types of boundaries for the given group G. In the CAT(0) case infinitely many pairwise non-homeomorphic boundaries have been produced [Wil05; Moo10]. Furthermore, at the moment we do not know whether boundaries of Helly groups are unique, that is, whether there exists a group acting on two Helly graphs whose associated boundaries are non-homeomorphic — see 9.Q1 for more detail.

Since CAT(0) groups and word hyperbolic groups are examples of groups acting geometrically on $\mathfrak{ccc}$-bicombable spaces, a natural question arises about relations between all types of boundaries. It is clear that if a word hyperbolic group acts geometrically on a space admitting a $\mathfrak{ccc}$ bicombing σ, then the boundary $\partial_\sigma X$ coincides with the Gromov boundary. For the CAT(0) spaces the situation is much more subtle, even in, otherwise restricted, case of CAT(0) cubical complexes. Here we have been able to obtain the following result in dimension 2.

Theorem V (Corollary 5.4). *Let X be a locally finite CAT(0) cube complex of dimension at most 2. Let σ^p be the convex bicombing on (X, d_p) for $p \in \{2, \infty\}$. Then the identity of X extends to a homeomorphism between $\overline{X}_{\sigma^2}$ and $\overline{X}_{\sigma^\infty}$, in particular the boundaries $\partial_{\sigma^2} X$ and $\partial_{\sigma^\infty} X$ are homeomorphic.*

In fact, we prove a more general result that the geodesics from both bicombings follow the same lines — see Theorem 5.2 — which in view of Proposition 4.8 implies the theorem above.

Our further results concern properties of the boundary analogous to the ones of CAT(0) boundary and the Gromov boundary. We were able to extend some results concerning the CAT(0) case to the more general setting of $\mathfrak{ccc}$-bicombable spaces.

The first of such results is the analogue of the result of Osajda–Świątkowski [OŚ15] (cf. also [Mor16]) concerning the existence of a particular metric on the boundary, and consequently, existence of a well-defined quasisymmetric structure.

Proposition VI (see Proposition 6.1(v)). *Let (X, d) be a complete metric space that admits an action of a group G and a $\mathfrak{ccc}$, G-equivariant bicombing σ. Then there exists a metric d_q on $\partial_\sigma X$ such that the extension of the action of each element of G to $\overline{X}_\sigma$ restricts to a quasisymmetry of $(\partial_\sigma X, d_q)$.*

Recall that in the word hyperbolic case an analogous quasisymmetric structure plays an important role in understanding the group. None such spectacular applications are known in the CAT(0) setting at the moment, due to a much looser connection between the boundary and the group. The metric from the above proposition has been used in research regarding the linearly-controlled dimension and the asymptotic dimension [Mor16]; we also use it, as a convenient tool in the further course of the text.

An important application of EZ-structures is a consequence of the relation between the topology of the boundary and algebraic properties of the group. Such relations are quite well understood in the word hyperbolic and CAT(0) case, see eg. [Bow98; Swe99; PS09]. We extend such studies to the case of groups acting geometrically on $\mathfrak{ccc}$-bicombable spaces. Here is a minor result in this direction.

Proposition VII (Proposition 7.1). *Let $G = G_1 *_Z G_2$ (with $G_1 \neq Z \neq G_2$), where Z is virtually $\mathbb{Z}$, act geometrically on a proper metric space (X, d_X) that admits a $\mathfrak{ccc}$, reversible, G-equivariant bicombing σ. Then there exists a separating pair of points in the boundary $\partial_\sigma X$.*

Another result analogous to the CAT(0) case is the following theorem about boundaries of groups containing abelian subgroups. In particular, besides boundaries of $\mathfrak{ccc}$-bicombable spaces coinciding with the Gromov or CAT(0) boundaries, this provides the first basic examples of EZ-structures for groups acting geometrically on $\mathfrak{ccc}$-bicombable spaces — such boundaries of free abelian groups are spheres, the same as in the CAT(0) case.

Proposition VIII (Proposition 7.2). *Let G be group that contains a free abelian subgroup $\mathbb{Z}^n \cong A < G$ and acts geometrically on a proper metric space X that admits a $\mathfrak{ccc}$, reversible, G-equivariant bicombing σ^X. Then $\partial_{\sigma^X} X$ contains a homeomorphic copy of S^{n-1}. Moreover, if A is of finite index in G, then $\partial_{\sigma^X} X \cong S^{n-1}$.*

The proof of an analogous statement in the CAT(0) setting easily follows from the fact that the minset of A, which is a convex subset of X, splits by the Flat Torus Theorem [BH99, II.7.2] as a metric product with one of the factors being an n-dimensional Euclidean space F, giving an n-dimensional flat F as a convex subset of the space X. However, in the context of spaces admitting $\mathfrak{ccc}$, reversible bicombings, even though flats exist, there is no such splitting nor such convexity of flats — see the first paragraph of the proof of the above proposition — which leads us to approaching the problem in a more elementary way.

In the case of a group G acting geometrically on a space X, the Alexander–Spanier cohomology (which is equivalent to the Čech cohomology) of the boundary describes some cohomological properties of the group — e.g. one has the Bestvina–Mess formula [Bes96] (in the torsion-free case), or in the case of groups acting $\mathfrak{ccc}$-bicombable spaces, one has the isomorphism between the group cohomology $H^{*+1}(G, \mathbb{Z}G)$ and the reduced cohomology of the boundary $\tilde{\bar{H}}^*(\partial_\sigma X)$ (see Remark 8.10). In this paper we present two $\mathfrak{ccc}$-counterparts of important results concerning the topology of CAT(0) boundaries of spaces acted upon cocompactly by a group, see [GO07; Ont05]. The meaning of the first of them is that the Alexander–Spanier cohomology behaves quite well with respect to the topology — it indicates the dimension of the boundary.

Theorem IX (Theorem 8.8). *Let X be a non-compact finite-dimensional proper metric space that admits a $\mathfrak{ccc}$ geodesic bicombing σ and a cocompact group action via isometries. Then the reduced Alexander–Spanier cohomology group $\tilde{\bar{H}}^{\dim \partial_\sigma X}(\partial_\sigma X)$ is non-zero.*

The second result is a pretty immediate consequence of the first one, but is very useful by itself. We say that a space X that admits a $\mathfrak{ccc}$ bicombing σ is *almost σ-geodesically complete* if the (the images of) σ-rays originating from some (equivalenty, any) point of

X are coarsely dense in X. This property allows one to perform a 'push-out' on various objects (e.g. paths or subspaces) from the space X 'towards infinity', in particular, to the boundary. It makes it possible to 'transfer' various reasonings between the space and its boundary hence and forth.

Theorem X (Theorem 8.1)**.** *Assume that X is a proper non-compact finite-dimensional geodesic metric space that admits a* $\mathfrak{ccc}$ *geodesic bicombing σ and a cocompact group action via isometries. Then X is almost σ-geodesically complete.*

In the course of this article, we prove also the following minor results that may be themselves of some interest.

In Proposition 1.3 we note that the construction from [BM19] of a conical, reversible bicombing out of a conical one, is equivariant, which leads to the fact that a finite group acting on a space that admits a conical bicombing has a fixed point.

In Proposition 4.6 we give a direct proof of the fact that local geodesics of a $\mathfrak{ccc}$ bicombing are global geodesics of this bicombing (such fact could have been deduced from a more general setting in [Mie17]).

In Proposition 5.11 we state and prove an observation that the boundary of a product of proper $\mathfrak{ccc}$-bicombable spaces X, Y, with respect to a naturally defined product bicombing, is the join of the boundaries of X and Y.

Organisation of the paper. In Section 1 we establish some notation, as well as provide and discuss some general notions used later in this article. We also prove Proposition 1.3.

In Section 2 we recall the construction of the boundary-compactification via geodesic rays as in [DL15], prove some preliminary facts about this construction, and in Subsection 2.1 we prove the existence of an EZ-structure (Theorem I) relying on this construction.

In Section 3 we recall the construction of the boundary-compactification via the Gelfand dual from [EW23] (in the, less general compared to [EW23], setting of metric spaces); in Subsection 3.1 we prove the existence of an EZ-structure (Theorem I) relying on this construction, and discuss the obtained EZ-structures; in Subsection 3.2 we prove that the EZ-structures obtained in Subsections 2.1 and 3.1 are equivalent (Proposition 3.6 and Corollary 3.7).

In Section 4 we prove the non-uniqueness of the boundary in the injective case — Theorem 4.1 (Thm. IV). During the preparations for this proof we prove Propositions 4.6 and 4.8.

In Section 5 we discuss the boundaries of CAT(0) cube complexes equipped with piecewise-ℓ^∞ metrics — justifying Theorem 5.2 and Corollary 5.4 (Thm. V) — and prove a proposition concerning the boundary of the product of spaces (Proposition 5.11).

In Section 6 we discuss a metric on the boundary that leads to a quasisymmetric structure on the boundary of $\mathfrak{ccc}$-bicombable spaces (Proposition 6.1, cf. Prop. VI).

In Section 7 we use the existence of axes and flats in bicombable spaces to prove Proposition 7.1 (Prop. VII) and Proposition 7.2 (Prop. VIII).

Section 8 concerns almost geodesic completeness for $\mathfrak{ccc}$-bicombable spaces: in Subsection 8.1 we prove some preparatory lemmas, and in Subsection 8.2 we prove Theorem 8.8 (Thm. IX) and Theorem 8.1 (Thm. X).

In Section 9 we collect and formulate some problems and open questions.

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1. Preliminaries

Let (X, d) be a metric space. Let $x \in X$, $r > 0$ and $A \subseteq X$. We denote by $B_X(x, r)$ (resp. $\overline{B}_X(x, r)$)) the open (resp. closed) ball of radius r around x, and put $B_X(A, r) := \bigcup_{x \in A} B_X(x, r)$ and $\overline{B}_X(A, r) := \bigcup_{x \in A} \overline{B}_X(x, r)$. We denote by id_X the identity (map) on the space X.

In this article we tend to omit the space-related sub- and superscripts of various objects when the space we are referring to is clear from the context.

Basic notions from basic coarse geometry. Let (X, d_X) and (Y, d_Y) be metric spaces.

For a subset $A \subseteq X$ and $C > 0$, we say that the set A is *C-dense* in X if $\overline{B}(A, C) = X$.

For a pair of maps $f, f' \colon X \to Y$, we say that they *are at finite distance (from each other)* whenever there exists a constant $C > 0$ such that for all $x \in X$ the inequality $d_Y(f(x), f'(x)) \leq C$ holds.

We say that a map $f \colon X \to Y$ is:

- *coarsely Lipschitz*, if there exists $C > 0$ such that for all $x, x' \in X$ the inequality $d_Y(f(x), f(x')) \leq C d_X(x, x') + C$ holds;
- a *quasi-isometric embedding*, if there exists $C > 0$ such that for all $x, x' \in X$ the inequality $C^{-1} d_X(x, x') - C \leq d_Y(f(x), f(x')) \leq C d_X(x, x') + C$ holds;

- a *quasi-isometry*, if it is coarsely Lipschitz and admits a *quasi-inverse*, i.e. a coarsely Lipschitz function $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are at finite distance from id_Y and id_X, respectively.

An elementary argument shows that any quasi-inverse of a quasi-isometry $X \to Y$ is automatically a quasi-isometry $Y \to X$. Another elementary argument shows that a function $f\colon X \to Y$ is a quasi-isometry iff it is a quasi-isometric embedding whose image is C-dense in Y for some $C > 0$.

Geodesic bicombings. We denote by $\operatorname{im}\gamma$ the image of the curve $\gamma\colon I \to X$, where $I \subseteq \mathbb{R}$ is a possibly infinite interval.

Definition 1.1. Let (X, d) be a geodesic metric space. A *(geodesic) bicombing* σ is a continuous function $\sigma\colon X \times X \times [0,1] \to X$ such that for each $x, y \in X$ the function $\sigma_{xy} := \sigma(x, y, \cdot)$ is a constant speed geodesic from x to y; we call each function σ_{xy} a σ*-geodesic*. We say that a bicombing σ is:

- *consistent*, if $\operatorname{im}\sigma_{\sigma_{xy}(s),\sigma_{xy}(t)} = \operatorname{im}\sigma_{xy}|_{[s,t]}$ for all $x, y \in X$ and $s, t \in [0,1]$ with $s < t$;
- *conical*, if $d(\sigma_{xy}(t), \sigma_{x'y'}(t)) \leq (1-t)d(x, x') + td(y, y')$ for all $x, y, x', y' \in X$, $t \in [0,1]$;
- *convex*, if the function $[0,1] \ni t \mapsto d(\sigma_{xx'}(t), \sigma_{yy'}(t))$ is convex for all $x, x', y, y' \in X$;
- *reversible*, if $\sigma_{xy}(t) = \sigma_{yx}(1-t)$ for all $x, y \in X$, $t \in [0,1]$;
- given an action of a group G on X and $g \in G$, *g-equivariant*, if $g\sigma_{xy} = \sigma_{gx,gy}$ for all $x, y \in X$, and *G-equivariant*, if it is g-equivariant for all $g \in G$.

We say that a subset $A \subseteq X$ is σ*-convex* if $\operatorname{im}\sigma_{xy} \subseteq A$ for any $x, y \in A$.

Remark 1.2. It is clear that every convex bicombing is conical. The converse holds for consistent bicombings — we need to show that given $x_1, y_1, x_2, y_2 \in X$, $0 \leq s < t \leq 1$ and $\alpha \in [0,1]$, we have that

$$\begin{aligned} &d(\sigma_{x_1,y_1}(\alpha s + (1-\alpha)t), \sigma_{x_2,y_2}(\alpha s + (1-\alpha)t)) \\ &\qquad\qquad \leq \alpha d(\sigma_{x_1,y_1}(s), \sigma_{x_2,y_2}(s)) + (1-\alpha)d(\sigma_{x_1,y_1}(t), \sigma_{x_2,y_2}(t)). \end{aligned}$$

Indeed, this inequality holds, since by consistency we have that $\sigma_{\sigma_{x_i,y_i}(s),\sigma_{x_i,y_i}(t)}(\alpha) = \sigma_{x_i,y_i}(\alpha s + (1-\alpha)t)$ for any $i \in \{1,2\}$ and $\alpha \in [0,1]$, so the inequality above reduces to an inequality asserted in the definition of conicality.

Bearing in mind the remark above, we call a geodesic bicombing a **ccc** *bicombing* if it is consistent, conical, and — therefore — convex.

We note that the following proposition follows from an appropriate application of methods and theorems due to Basso, Descombes, Lang and Miesch.

Proposition 1.3. *Assume that G is a finite group acting of a metric space X via isometries, and that X admits a conical, G-equivariant geodesic bicombing σ. Then the action of G on X has a fixed point.*

Proof. Basso and Miesch in [BM19, Proposition 1.3] used the following procedure to produce from a conical bicombing σ a 'midpoint map' $m\colon X \times X \to X$, and then a conical, reversible bicombing σ^R. Let $x, y \in X$. Define $x_0 := x$, $y_0 := y$, and $x_{n+1} := \sigma_{x_n,y_n}(1/2)$, $y_{n+1} := \sigma_{y_n,x_n}(1/2)$ for $n \in \mathbb{N}$. Then the sequences (x_n) and (y_n) are convergent to the same limit; define $m(x, y)$ to be this limit. Finally, $\sigma^R_{xy}(t) := m(\sigma_{xy}(t), \sigma_{yx}(1-t))$ defines a conical, reversible bicombing on X. It is easy to check that, since σ is G-equivariant, the constructions of the sequences x_n and y_n, of the map m, and of the bicombing σ^R are all also G-equivariant.

Descombes, Lang and Basso [DL16; Bas18], building on [EH99; Nav13], introduced a construction of barycentre maps $\mathrm{bar}_n\colon X^n \to X$ for $n \in \mathbb{N}$ for spaces admitting conical bicombings. If the input bicombing is additionally reversible, then the functions bar_n are invariant under permuting their arguments (see [DL16, Theorem 4.1(2)] or [Bas18, Proposition 3.4(4)]) and, if the input bicombing is G-equivariant, then the barycentre maps are also G-equivariant (see [DL16, Theorem 4.1(3)] or [Bas18, around formula (3.5)]). It is now easy to see that the barycentre of any orbit of G constructed using the bicombing σ^R is a fixed point of the action of G on X. □

Euclidean retracts and absolute retracts. A space is a *Euclidean retract (ER)* if it can be embedded in some Euclidean space as its retract.

A space is an *absolute retract (AR)* if, whenever it is embedded into a metric space Y as a closed subspace A, the subspace A is a retract of the space Y.

A locally compact, separable metric space is an ER iff it is finite-dimensional, contractible and locally contractible. We provide an outline of the proof of this characterisation in the following proposition, for completess.

Proposition 1.4. *Let X be a metric space. Then:*

(i) if X locally compact and separable, then X is an ER iff X is a finite-dimensional AR;

(ii) if X is finite-dimensional, then X is an AR iff X is contractible and locally contractible.

Proof. (i), the $\Longrightarrow$ implication. Let $\imath\colon X \to \mathbb{R}^n$ be an embedding such that the set $\imath(X)$ is a retract of the space $\mathbb{R}^n$. Since X is locally compact, there exists an open subset $U \subseteq \mathbb{R}^n$ such that $\imath(X)$ is contained in U as a closed subset. Then a standard trick of taking the product of the map $\imath$ with the map $X \ni x \mapsto d(\imath(x), \mathbb{R}^n \setminus U)^{-1} \in \mathbb{R}$ gives an embedding of X into $\mathbb{R}^{n+1}$ as a closed subset. Therefore $\dim X \leq n+1$, see [Eng78, Theorem 3.1.4].

Assume that we have an embedding $\jmath$ of X into a metric space Y such that $\jmath(X)$ is closed in Y. By the Tietze's Extension Theorem, the homeomorphism $\imath \circ \jmath^{-1} \colon \jmath(X) \to \imath(X) \subseteq \mathbb{R}^n$ can be extended to a map $Y \to \mathbb{R}^n$. Composing this map with the retraction of $\mathbb{R}^n$ onto $\imath(X)$, and then with the inverse of the homeomorphism $\imath \circ \jmath^{-1}$ gives a retraction of Y onto $\jmath(X)$.

(i), THE $\Longleftarrow$ IMPLICATION. By the Embedding Theorem, see [Eng78, Theorem 1.11.4], the separable metric space X embeds into a Euclidean space $\mathbb{R}^n$. As in the first paragraph of the proof, by local compactness of X, the space X can be embedded into $\mathbb{R}^{n+1}$ as a closed subset. Then the AR-ness of X gives the desired retraction.

(ii) This is exactly the equivalence of (a) and (b) in [Hu65, Theorem V.11.1]. □

For more information on the topic one may refer to [Hu65] or [vMi89], or to the concise introduction in [GM19, Section 2].

EZ-structures. A compact subset Z of a compact space $\mathfrak{X}$ is a *Z-set* in $\mathfrak{X}$ if there exists a homotopy $\{H_t \colon \mathfrak{X} \to \mathfrak{X} : t \in [0,1]\}$ such that $H_0 = \mathrm{id}_{\mathfrak{X}}$ and $H_t(\mathfrak{X}) \cap Z = \emptyset$ for any $t > 0$.

A sequence K_n of subsets of a topological space X is called a *null-sequence* if for every open cover $\mathcal{U}$ of X, for all but finitely many n the set K_n is *$\mathcal{U}$-small*, that is, there exists a set $U \in \mathcal{U}$ such that $K_n \subseteq U$.

Definition 1.5. Let $Z \subseteq \mathfrak{X}$ and G act geometrically on $\mathfrak{X} \setminus Z$. The pair $(\mathfrak{X}, Z)$ is an *EZ-structure* for G if $\mathfrak{X}$ an ER, Z is a Z-set in $\mathfrak{X}$, $(gK)_{g \in G}$ is a null-sequence in $\mathfrak{X} \setminus Z$ for every compact set $K \subseteq \mathfrak{X} \setminus Z$, and the action of G on $\mathfrak{X} \setminus Z$ extends to an action by homeomorphisms on $\mathfrak{X}$.

The notion of the Z-structure has been introduced by Bestvina in [Bes96], and its equivariant version, the EZ-structure, has been introduced in [FL05]. These notions have later been studied, precised and generalised in e.g. [Dra06; OP09]. We note that there exist several variations of the definition of the EZ-structure, mainly differing with the above by altering the ER-ness assumption to an AR-ness assumption, or by altering the condition that the group action is geometric to other conditions imposed on the group action. In the definition above we followed the last of the mentioned articles.

Gluings. Further in this article we often encounter the following gluing setting. We are given a family of metric spaces (X_i, d_i) for $i \in I$ and a surjective map $\pi \colon \bigsqcup_{i \in I} X_i \to X$ such that π restricted to each of the X_i is one-to-one, and such that for each $i, j \in I$ the map $\pi|_{X_j}^{-1} \circ \pi|_{X_i}$ induces an isometry from $(\pi|_{X_i}^{-1}(\pi(X_i) \cap \pi(X_j)), d_i)$ to $(\pi|_{X_j}^{-1}(\pi(X_i) \cap \pi(X_j)), d_j)$. Such a map π is called a *gluing map.* We construct the following gluing pseudometric on X, which often turns out to be a metric in our settings. Let $d(x, x') := \inf\{\operatorname{len}(P) : P \text{ is a } (x, x')\text{–gluing path}\}$, where an (x, x')*–gluing path* is a sequence of points $P = (x = x_0, \ldots, x_n = x')$ such that $n \in \mathbb{N}$ and for each $0 \le j \le n-1$ there exists $i_j \in I$ such that $x_j, x_{j+1} \in \pi(X_{i_j})$, and we define its length as $\operatorname{len}(P) := \sum_{i=0}^{n-1} d_{X_{i_j}}(x_j, x_{j+1})$. We often do not distinguish the set X_i from $\pi(X_i)$ and treat it as a subset of X.

2. Boundary via geodesic rays

We begin with a brief summary of the construction in [DL15, Section 5].

Let (X, d) be a complete geodesic metric space with a ccc bicombing σ. A σ-ray in X is an isometric embedding $\xi \colon [0, \infty) \to X$ such that $\operatorname{im} \sigma_{\xi(0)\xi(t)} = \operatorname{im} \xi|_{[0,t]}$ for every $t \ge 0$. Two σ-rays ξ, ζ are asymptotic if their images are at finite Hausdorff distance, equivalently, $\sup_{t \ge 0} d(\xi(t), \zeta(t)) < \infty$, and we denote by $[\xi]$ the set of σ-rays asymptotic to the σ-ray ξ.

The boundary $\partial_\sigma X$ is a topological space whose underlying set consists of the set of classes of asymptotic σ-rays. By [DL15, Proposition 5.2], for each basepoint $o \in X$, every class $\bar{x} \in \partial_\sigma X$ has a unique representative $\varrho_{o,\bar{x}}$ that originates in o, therefore the boundary $\partial_\sigma X$ may be identified with the set of σ-rays originating at some arbitrary fixed point o. We often consider the set $\overline{X}_\sigma := X \cup \partial_\sigma X$, and extend the definition of ϱ to X by defining functions $\varrho_{o,x} \colon [0, \infty) \to X$ for $x \in X$ by stopping after reaching x, i.e. $\varrho_{o,x}(t) = \sigma_{ox}(\min(t/d(o, x), 1))$ for any $o, x \in X$, and identify point $x \in X$ with $\varrho_{o,x}$.

The topology on $\overline{X}_\sigma$ is given by the following base: for a fixed basepoint $o \in X$, take $\{U_o(\bar{x}, t, \epsilon) : \bar{x} \in \overline{X}_\sigma, t \ge 0, \epsilon > 0\}$ where $U_o(\bar{x}, t, \epsilon) = \{\bar{y} \in \overline{X}_\sigma : d(\varrho_{o,\bar{y}}(t), \varrho_{o,\bar{x}}(t)) < \epsilon\}$. One can show that the resulting topology does not depend on the choice of the basepoint o, the topology on $\overline{X}_\sigma$ extends the topology on X — see [DL15, Lemma 5.3 and above] — and the topology on $\partial_\sigma X$ has a base $\{U_o(\bar{x}, t, \epsilon) \cap \partial_\sigma X : \bar{x} \in \partial_\sigma X, t \ge 0, \epsilon > 0\}$. One may observe that the topology coincides with the topology arising from viewing $\overline{X}_\sigma$ as the inverse limit of the system $(\{X_R : R \ge 0\}, \{\pi_r^R \colon X_R \to X_r : 0 \le r \le R\})$ where $X_R = \overline{B}(o, R) (= \bigcup\{\varrho_{o,x}([0, R]) : x \in X\})$ and $\pi_r^R(\varrho_{o,x}(R)) = \varrho_{o,x}(r)$ for all $x \in X$ (which is well-defined by consistency of σ). Thus $\overline{X}_\sigma$ admits a metric

$$d_o(\bar{x}, \bar{y}) := \sum_{n=1}^{\infty} 2^{-n} \min\bigl(d(\varrho_{o,\bar{x}}(n), \varrho_{o,\bar{y}}(n)), 1\bigr), \tag{2-1}$$

arising from viewing $\overline{X}_\sigma$ as the subspace $\{(x_n)_{n \in \mathbb{N}} : x_n \in X_n, \pi_n^{n+1}(x_{n+1}) = x_n\} \subseteq \prod X_n$

(we note that this metric is in the same spirit as the metric D_o introduced in [DL15]), and the space $\overline{X}_\sigma$ is compact iff X is proper. Note that for all $r > 0$ the map given by $\pi_r(\bar{x}) = \varrho_{o,\bar{x}}(r)$ corresponds to the projection map $\pi_r \colon \overline{X}_\sigma \to X_r$ from the definition of an inverse limit.

We finish this introduction with three useful observations. Proposition 2.2 will be used in the proof of Theorem I in this section, and Proposition 2.1 and Proposition 2.4 will be used in other parts of this article.

Proposition 2.1. *Assume that σ is a* ccc *bicombing on a complete metric space (X, d) and ζ, η are σ-rays.*

(i) The function $D(t) := d(\zeta(t), \eta(t))$ is convex. In particular, if ζ and η are asymptotic, then D is non-increasing.

(ii) Let Ξ be a family of σ-rays originating from a common point and such that the set $\{[\xi] : \xi \in \Xi\}$ is compact in $\partial_\sigma X$. If

$$(\exists D \geq 0)(\forall r > 0)(\exists t \geq r, \xi \in \Xi)(d(\zeta(t), \operatorname{im} \xi) \leq D),$$

then ζ is asymptotic to some σ-ray from Ξ.

(iii) The σ-rays ζ and η are asymptotic iff $(\exists D \geq 0)(\forall r > 0)(\exists t \geq r)(d(\zeta(t), \operatorname{im} \eta) \leq D)$.

Proof. (i) Follows from convexity of σ.

(ii) Assume that there exists $D \geq 0$ and sequences $t_n \to \infty$, $s_n \geq 0$ and $\xi_n \in \Xi$, such that $d(\zeta(t_n), \xi_n(s_n)) \leq D$. Since ζ and ξ_n are geodesic rays,

$$\begin{aligned} |t_n - s_n| &= |d(\zeta(t_n), \zeta(0)) - d(\xi_n(0), \xi_n(s_n))| \\ &\leq d(\zeta(t_n), \xi_n(s_n)) + d(\zeta(0), \xi_n(0)) \leq D + d(\zeta(0), \xi_n(0)), \end{aligned}$$

and $d(\zeta(t_n), \xi_n(t_n)) \leq 2D + d(\zeta(0), \xi_n(0))$ for any $n \in \mathbb{N}$. By compactness, one may choose a subsequence n_k such that $[\xi_{n_k}]$ converge to $[\xi]$ for some $\xi \in \Xi$. Then, for any k and $t \leq t_{n_k}$, by convexity of σ we have that

$$\begin{aligned} d(\zeta(t), \xi(t)) &\leq d(\zeta(t), \xi_{n_k}(t)) + d(\xi_{n_k}(t), \xi(t)) \\ &\leq \max\bigl(d(\zeta(0), \xi_{n_k}(0)), d(\zeta(t_{n_k}), \xi_{n_k}(t_{n_k}))\bigr) + d(\xi_{n_k}(t), \xi(t)) \\ &\leq 2D + d(\zeta(0), \xi_{n_k}(0)) + d(\xi_{n_k}(t), \xi(t)). \end{aligned}$$

Therefore, since the σ-rays from Ξ are based in one point, and $t_{n_k} \to \infty$, and $\xi_{n_k}(t) \to \xi(t)$ for all $t \geq 0$, the σ-rays ξ and ζ are asymptotic.

(iii) The $\Longrightarrow$ implication follows from (i). The $\Longleftarrow$ implication follows from (ii) with $\Xi = \{\eta\}$. $\square$

The following builds on [DL15, inequality (5.2)].

Proposition 2.2. *Let (X,d) be a metric space with a* ccc *bicombing σ, and $x,y,o \in X$, $r>0$ be such that* $\max(d(o,x),d(o,y)) \geq r$. *Then* $d(\varrho_{o,x}(r),\varrho_{o,y}(r)) \leq 2\cdot d(x,y)\cdot r/d(o,x)$.

Proof. When $d(o,x) \geq d(o,y)$ and $d(o,x) \geq r$, the inequality is (almost) [DL15, inequality (5.2)] and can be justified as follows: note that $\varrho_{o,x}(r) = \sigma_{o,x}(r/d(o,x))$, and denote $y_{\sim r} = \sigma_{o,y}(r/d(o,x))$; then

$$\begin{aligned}
d(\varrho_{o,x}(r),\varrho_{o,y}(r)) &\leq d(\varrho_{o,x}(r),y_{\sim r}) + d(y_{\sim r},\varrho_{o,y}(r)) \\
&\leq \frac{r}{d(o,x)}d(x,y) + d(o,\varrho_{o,y}(r)) - d(o,y_{\sim r}) \\
&= \frac{r}{d(o,x)}d(x,y) + \min(r,d(o,y)) - \frac{r}{d(o,x)}d(o,y) \\
&\leq \frac{r}{d(o,x)}d(x,y) + r\frac{d(o,x)-d(o,y)}{d(o,x)} \leq \frac{r}{d(o,x)}d(x,y) + r\frac{d(x,y)}{d(o,x)} = 2r\frac{d(x,y)}{d(o,x)}.
\end{aligned}$$

If $d(o,y) \geq d(o,x)$ and $d(o,y) \geq r$, the above gives that

$$d(\varrho_{o,x}(r),\varrho_{o,y}(r)) \leq 2r\,d(x,y)/d(o,y) \leq 2r\,d(x,y)/d(o,x). \qquad \square$$

Definition 2.3. Let $o \in X$. We define a map $\ell_o\colon \overline{X}_\sigma \to [0,\infty]$ by putting $\ell_o(x) = d(o,x)$ for $x \in X$ and $\ell_o(\bar{x}) = \infty$ for $\bar{x} \in \partial_\sigma X$; and define an exponential map $\exp_o\colon \overline{X}_\sigma \times [0,\infty] \to \overline{X}_\sigma$ by $\exp_o(\bar{x},t) = \varrho_{o,\bar{x}}(t)$ for $\bar{x} \in \overline{X}_\sigma$ and $t<\infty$, and $\exp_o(\bar{x},\infty) = \bar{x}$.

Proposition 2.4. *Let (X,d) be a complete metric space with a* ccc *bicombing σ. Then the maps ℓ_o and $\exp_o$ are continuous for any basepoint $o \in X$.*

Proof. The function ℓ_o is continuous since it is continuous on X as the metric d is continuous with respect to itself, and for $\bar{x} \in \partial_\sigma X$ we have that $\ell_o(U_o(\bar{x},R,\delta)) \subseteq [R-\delta,\infty]$.

For the proof of continuity of $\exp_o$, assume that we have sequences $(\bar{x}_n) \subseteq \overline{X}_\sigma$ convergent to $\bar{x} \in \overline{X}_\sigma$ and $(t_n) \subseteq [0,\infty]$ convergent to $t \in [0,\infty]$. If $t=\infty$, then for any $s<\infty$ we have for sufficiently large n that $\varrho_{o,\exp_o(\bar{x}_n,t_n)}(s) = \varrho_{o,\bar{x}_n}(s)$, as $t_n \to t = \infty$; the latter converges to $\varrho_{o,\bar{x}}(s) = \varrho_{o,\exp_o(\bar{x},\infty)}(s)$, as $\bar{x}_n \to \bar{x}$; therefore $\exp_o(\bar{x}_n,t_n)$ converges to $\exp_o(\bar{x},\infty)$. Assume that $t<\infty$. If $\bar{x} \in X$, then $\bar{x}_n \in X$ for sufficiently large n, so we can write by continuity of σ that

$$\exp_o(\bar{x}_n,t_n) = \sigma\left(o,\bar{x}_n,\frac{\min(t_n,d(o,\bar{x}_n))}{d(o,\bar{x}_n)}\right) \longrightarrow \sigma\left(o,\bar{x},\frac{\min(t,d(o,\bar{x}))}{d(o,\bar{x})}\right) = \exp_o(\bar{x},t).$$

Assume that $\bar{x} \in \partial_\sigma X$. For sufficiently large n we have that $t_n \leq R$ (and thus $t \leq R$) for some $R<\infty$. Then, by consistency of σ, we have for sufficiently large n that $\exp_o(\bar{x}_n,t_n) = \exp_o(\exp_o(\bar{x}_n,R),t_n)$. Since $\exp_o(\bar{x}_n,R)$ converges to $\exp_o(\bar{x},R)$, as $\bar{x}_n$

converges to $\bar{x}$, we can use the previous case to conclude that $\exp_o(\bar{x}_n, t_n)$ converges to $\exp_o(\exp_o(\bar{x}, R), t) = \exp_o(\bar{x}, t)$. $\square$

2.1. The EZ-structure

Below we describe how the above construction of boundary may be used to construct an EZ-structure.

Proof. (of Theorem I) We show that $(\overline{X}_\sigma, \partial_\sigma X)$ is an EZ-structure for G (see Definition 1.5). Fix any basepoint $o \in X$.

The boundary $\partial_\sigma X$ is a Z-set in $\overline{X}_\sigma$ by [DL15, Theorem 1.4]. Fix any homotopy $\{H_t : t \in [0,1]\}$ from the definition of Z-set.

Now we show that the compact space $\overline{X}_\sigma$ is an ER, using the characterisation from Proposition 1.4. The space $\overline{X}_\sigma$ is contractible and locally contractible by [DL15, Theorem 1.4]. The fact that the dimension of $\overline{X}_\sigma$ is not greater than the dimension of X, therefore finite, is a standard task in topology and can be justified as follows. Let $\mathcal{U}$ be an open cover of $\overline{X}_\sigma$. Let λ be the Lebesgue number of $\mathcal{U}$. Since X is finite-dimensional, there exists an open cover $\mathcal{V}$ of X consisting of sets of d_o-diameter at most $\lambda/3$ and having empty intersections of each of its subfamilies of cardinality $\dim X + 2$. By compactness of $\overline{X}_\sigma$, there exists $t_0 > 0$ such that for all $x \in \overline{X}_\sigma$ we have $d_o(H_{t_0}(x), x) < \lambda/3$. It follows that $\{H_{t_0}^{-1}(V) : V \in \mathcal{V}\}$ is an open cover of $\overline{X}_\sigma$, is a refinement of $\mathcal{U}$, and has the property that intersections of any of its $\dim X + 2$ sets are empty. We note that a similar proof of the fact that $\dim \overline{X}_\sigma \leq \dim X$ when $\dim X < \infty$ is also present in the proof of [DL15, Theorem 1.4], where X is not assumed to be proper and a suitable value t_0 is stated explicitly using the metric D_o, or in the proof of [EW23, Theorem 7.10].

We claim that for any compact set $K \subseteq X$ and any open cover $\mathcal{U}$ of $\overline{X}_\sigma$ all but finitely many of the translates $(gK)_{g\in G}$ are $\mathcal{U}$-small. By properness of X, $K \subseteq B(o,R)$ for some $R \geq 0$. By compactness, let $\{U_o(\bar{x}_i, t_i, \epsilon_i) \cap \partial_\sigma X : 1 \leq i \leq n\}$ for some $\bar{x}_i \in \partial_\sigma X$, $t_i \geq 0$, $\epsilon_i > 0$ be a finite cover of $\partial_\sigma X$ such that each element of $\{U_o(\bar{x}_i, t_i, 2\epsilon_i) : 1 \leq i \leq n\}$ is contained in some element of $\mathcal{U}$. Since $\overline{X}_\sigma \setminus \bigcup U_o(\bar{x}_i, t_i, \epsilon_i)$ is a compact subset of X, it is contained in a ball $B(o, R_0)$ for some $R_0 \geq 0$. Take $g \in G$ such that $d(o, go) > R_0$. Then there exists some $1 \leq k \leq n$ such that $go \in U_o(\bar{x}_k, t_k, \epsilon_k)$. If, additionally, $d(o, go) > t_k$ and $2Rt_k/d(o,go) \leq \epsilon_k$, then, by Proposition 2.2 with $r = t_k$ and $x = go$,

$$gK \subseteq gB(o,R) = B(go,R) \subseteq U_o(go, t_k, 2Rt_k/d(o,go)) \subseteq U_o(go, t_k, \epsilon_k) \subseteq U_o(\bar{x}_k, t_k, 2\epsilon_k),$$

which is contained in some set from the family $\mathcal{U}$. Since G acts on X properly, all but finitely many $g \in G$ satisfy $d(o, go) > \max\big(\{R_0\} \cup \{t_i : 1 \leq i \leq n\} \cup \{2Rt_i/\epsilon_i : 1 \leq i \leq n\}\big)$, and the claim follows.

We extend the action of the group G on X to $\overline{X}_\sigma$ by defining $g\bar{x} := [t \mapsto g\varrho_{o,\bar{x}}(t)]$ for all $g \in G$, which is well-defined since G acts via isometries and σ is G-equivariant. It remains to show that such an extended action is an action via homeomorphisms. By G-equivariance of σ, for any $g \in G$, $\bar{x} \in \overline{X}_\sigma$, $t \geq 0$ and $\epsilon > 0$ we have that $gU_o(\bar{x}, t, \epsilon) = U_{go}(g\bar{x}, t, \epsilon)$. Therefore g maps a base of the topology of $\overline{X}_\sigma$ arising from using the basepoint o to another base, resulting from taking go as the basepoint. □

3. Boundary via Gelfand dual

In this section we discuss the EZ-structure resulting from the construction of the boundary introduced by Engel and Wulff in [EW23]. Then we prove equivalence of this EZ-structure with the one constructed in Section 2. We try to give a more elementary treatment of the subject compared to the original paper.

Below we briefly describe the construction of the boundary from [EW23], which applies to the so called coarse spaces, in our less general metric setting (see [EW23, Example 2.3]). Let (X, d) be a metric space and fix a basepoint $o \in X$. We say that a function $\Sigma\colon X \times \mathbb{N} \to X$, where we often use the notation $\Sigma_n(x) = \Sigma(x, n)$, is a *combing*, and say that (X, Σ) is a *combed space*, if the following are satisfied:

- $\Sigma_n(o) = o = \Sigma_0(x)$ for all $x \in X$, $n \in \mathbb{N}$;
- for all $R \geq 0$ there exists $N \in \mathbb{N}$ such that $\Sigma_n(x) = x$ for all $n \geq N$ and $x \in \overline{B}(o, R)$;
- for all $D > 0$ there exists C such that for all $x, x' \in X$ with $d(x, x') \leq D$, and for all $n, n' \in \mathbb{N}$ with $|n - n'| \leq D$, it holds that $d(\Sigma(x, n), \Sigma(x', n')) \leq C$.

(An example of a combing, related to Section 2, is $\Sigma(x, n) := \varrho_{o,x}(n)$.) We say that a combing Σ is *coherent* if there exists $R \geq 0$ such that $d(\Sigma_n(x), \Sigma_m(\Sigma_n(x))) \leq R$ for all $m \leq n \in \mathbb{N}$ and $x \in X$. A map $\alpha\colon X \to Y$ is a morphism between combed spaces (X, Σ_X) and (Y, Σ_Y) if the function $(x, n) \mapsto d(\alpha(\Sigma_X(x, n)), \Sigma_Y(\alpha(x), n))$ is bounded (see [EW23, Remark 2.5(a)]). Morphisms, coarsely Lipschitz maps, and being at bounded distance work with each other in the following way.

Proposition 3.1. *Assume that metric spaces $(X, d_X), (Y, d_Y)$ admit combings Σ_X, Σ_Y, respectively.*

(i) Let $f\colon X \to Y$, $g\colon Y \to X$ be coarsely Lipschitz such that $f \circ g$ is at finite distance from the identity of Y. Then, if f is a morphism from (X, Σ_X) to (Y, Σ_Y), then g is a morphism from (Y, Σ_Y) to (X, Σ_X).

(ii) Assume that $f'\colon X \to Y$ is at finite distance from a morphism f from (X, Σ_X) to (Y, Σ_Y). Then f' is also a morphism from (X, Σ_X) to (Y, Σ_Y).

Proof. (i) Since $f\colon X \to Y$ is coarsely Lipschitz, there exists a constant C such that for any $y \in Y$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} d_X\big(g(\Sigma_Y(y,n)), \Sigma_X(g(y),n)\big) &\le C d_Y\big(f(g(\Sigma_Y(y,n))), f(\Sigma_X(g(y),n))\big) + C \\ &\le C\Big(d_Y\big(f(g(\Sigma_Y(y,n))), \Sigma_Y(y,n)\big) + d_Y\big(\Sigma_Y(y,n), \Sigma_Y(f(g(y))),n\big) \\ &\qquad + d_Y\big(\Sigma_Y(f(g(y)),n), f(\Sigma_X(g(y),n))\big)\Big) + C. \end{aligned}$$

The claim follows, as each of 3 terms in the parentheses is bounded by a constant independent of y and n: the first one by the fact that $f \circ g$ is at finite distance from the identity of Y; the second one by the fact that $f \circ g$ is at finite distance from the identity of Y, and the third ($\bullet$) in the definition of a combing; the last one by the fact that f is a morphism between (X, Σ_X) and (Y, Σ_Y).

(ii) For any $x \in X$, $n \in \mathbb{N}$ we have the following inequality:

$$\begin{aligned} d_Y\big(f'(\Sigma_X(x,n)), \Sigma_Y(f'(x),n)\big) &\le d_Y\big(f'(\Sigma_X(x,n)), f(\Sigma_X(x,n))\big) \\ &\quad + d_Y\big(f(\Sigma_X(x,n)), \Sigma_Y(f(x),n)\big) + d_Y\big(\Sigma_Y(f(x),n), \Sigma_Y(f'(x),n)\big). \end{aligned}$$

The claim follows, as each of these 3 terms is bounded by a constant independent of x and n: the first one by the fact that the functions f and f' are at finite distance from each other; the second one by the fact that f is a morphism; the last one by the fact the functions f and f' are at finite distance from each other, and the third ($\bullet$) in the definition of combing. $\square$

Given a coherent combing Σ on a proper metric space X, one can construct the compactification $\overline{X}^{\Sigma}$ (technically, the construction below applies in a more general setting of proper combings, see [EW23, Definition 2.6 and Lemma 2.7]) as follows. Define $C_\Sigma(X)$ to be the C^*-algebra of all continuous, bounded functions $f\colon X \to \mathbb{C}$ such that $f \circ \Sigma_n \to f$ (in the supremum metric) and f has bounded variation, i.e. for all $R > 0$ the variation function $\mathrm{Var}_R(f)(x) := \sup\{|f(y) - f(x)| : y \in \overline{B}(x,R)\}$ is below any $\epsilon > 0$ outside of a compact set $K(\epsilon, f, R)$. We consider the Gelfand dual $\overline{X}^{\Sigma}$ of $C_\Sigma(X)$ — the space of non-zero multiplicative $\mathbb{C}$-linear functionals on the algebra $C_\Sigma(X)$ (these properties imply continuity (with norm 1), see [Lin01, Theorem 1.3.2(i)]) with the weak-* topology. Taking the kernel of such functionals gives an identification of the underlying set of $\overline{X}^{\Sigma}$ with the set of maximal (proper) ideals in $C_\Sigma(X)$, see [Lin01, Theorem 1.3.2(ii)]. The map $X \ni x \mapsto \delta_x \in \overline{X}^{\Sigma}$, where δ_x is the evaluation at x, is a well-defined homeomorphic embedding (for more details, see the proof of Proposition 3.6).

3.1. The EZ-structure

Below we show a proof of existence of an EZ-structure, relying on the construction presented previously in this section. Next, we describe and compare the EZ-structures obtained in this proof.

Proof. (of Theorem I) Fix a basepoint $o \in X$. By the Švarc–Milnor lemma, the group G is finitely generated, and the map $\alpha\colon G \to X$ given by the formula $\alpha(g) = go$ is a G-equivariant quasi-isometry; therefore the map $\Sigma(x,n) := \varrho_{o,x}(n)$, which is clearly a combing of X, can be moved to a combing Σ_G of G using α as follows: let $\beta\colon X \to G$ be a quasi-inverse of α and put $\Sigma_G(g,n) := \beta(\Sigma(\alpha(g),n))$ (the basepoint for Σ_G is $\beta(o)$). We list the assumptions of [EW23, Theorem 7.10 and Remark 7.15], whose conclusion is that G admits an EZ-structure (the yet-unexplained terms appearing in this list are addressed when needed later in the proof, in a form adapted to our metric setting): X is a Euclidean retract, the space X and the Rips complex $\mathrm{Rips}(G)$ are G-equivariantly homotopy equivalent, and the combing Σ_G is coherent, expanding and G-coarsely equivariant. We shall show that all these conditions are satisfied.

By the characterisation from Proposition 1.4, the space X is an ER, as it is locally compact, finite-dimensional (by assumption), contractible (by the existence of bicombing) and locally contractible (the balls in X are σ-convex because σ is conical).

Now we check that the space X and the Rips complex of G are G-equivariantly homotopy equivalent. The Rips complex $\mathrm{Rips}(G)$ is defined in [EW23, Definition 4.1 and Example 4.4] as the (increasing) union of the Rips complexes $\mathrm{Rips}_n(G)$ for $n \in \mathbb{N}$, where the complex $\mathrm{Rips}_C(G)$ for $C > 0$ is the typically discussed Rips complex, i.e. a finite subset T of elements of G spans a simplex of $\mathrm{Rips}_C(G)$ iff every two elements of T are at distance at most C. (This way every finite tuple of elements of G spans a simplex of $\mathrm{Rips}(G)$, and $\mathrm{Rips}(G)$ is equipped with the direct limit topology induced by the inclusions $\mathrm{Rips}_n(G) \subseteq \mathrm{Rips}(G)$.) In view of [Lüc05, Lemma 3.3], it is sufficient to show that $\mathrm{Rips}(G)$ is a model of $\underline{E}G$ and X is a model of $\underline{J}G$ (a 'numerable' version of $\underline{E}G$, see [Lüc05, Definition 2.3]).

We use the homotopy characterisation of $\underline{E}G$, see [Lüc05, Theorem 1.9(ii)], to show that the Rips complex $\mathrm{Rips}(G)$ is a model for $\underline{E}G$. Namely, we check that the stabiliser of each element of $\mathrm{Rips}(G)$ under the action of G is finite, and that for each finite subgroup H of G the set $\mathrm{Rips}(G)^H$ of fixed points of the action of H on $\mathrm{Rips}(G)$ is contractible. Each element $x \in \mathrm{Rips}(G)$ is contained in the interior of a simplex spanned by some elements $g_1, \dots, g_m \in G$, therefore every element of the stabiliser $\mathrm{Stab}_G(x)$ of x fixes setwise the set $\{g_i : 1 \leq i \leq m\}$, so $|\mathrm{Stab}_G(x)| \leq m < \infty$. For each finite subgroup H of

G the set $\mathrm{Rips}(G)^H$ is non-empty, as it contains the barycentre of the simplex spanned by the elements of H. It is easy to verify that for each pair of points $x, y \in \mathrm{Rips}(G)^H$, the set $\mathrm{Rips}(G)^H$ also contains the linear segment in $\mathrm{Rips}(G)$ between the points x and y. Therefore, after fixing any point $x \in \mathrm{Rips}(G)^H$, going along linear segments in $\mathrm{Rips}(G)$ towards x gives a homotopy $c_x\colon \mathrm{Rips}(G) \times [0,1] \to \mathrm{Rips}(G)$, which restrict to a homotopy $\mathrm{Rips}(G)^H \times [0,1] \to \mathrm{Rips}(G)^H$, which contracts the set $\mathrm{Rips}(G)^H$ to the point x. The continuity of c_x may be justified by the following argument. Let Δ_x be a finite subset of G such that x belongs to a simplex spanned by the elements of Δ_x. Consider a point $y \in \mathrm{Rips}(G)$ contained in a simplex spanned by the elements of a finite subset Δ_y of G. We now check continuity of c_x at all points of the form (y, t) for $t \in [0,1]$. Let U_y be the union of interiors of all of the simplices of $\mathrm{Rips}(G)$ that contain y. Then the set U_y is an open neighbourhood of x in $\mathrm{Rips}(G)$. Let $C \in \mathbb{N}$ be greater than any distance between a pair of points from the set $\Delta_x \cup \Delta_y$. Then, for each $n \in \mathbb{N}$ the contraction c_x restricts to a map $(U_y \cap \mathrm{Rips}_n(G)) \times [0,1] \to \mathrm{Rips}_{C+n}(G)$, which is continuous. Passing to the limit with n gives continuity of c_x on $U_y \times [0,1]$.

We use the homotopy characterisation of $\underline{J}G$, see [Lüc05, Theorem 2.5(ii) and Definition 2.1], to show that X is a model of $\underline{J}G$. Namely, we need to show that (i) X admits an open cover $\{U_i : i \in I\}$ such that (a) each of the sets U_i is G-invariant and admits a G-equivariant map $U_i \to G/G_i$ for some finite subgroup G_i of G, and (b) there exists a locally finite partition of unity on X via G-invariant functions, subordinate to $\{U_i : i \in I\}$; (ii) each finite subgroup of G has a fixed point in X; and (iii) the projection maps $X \times X \to X$ onto the first coordinate and onto the second coordinate are homotopic via a G-equivariant homotopy. Regarding property (i), first observe that properness of the action of G gives that for each point $x \in X$ there exists a number $r_x > 0$ such that for all $g \in G$ either $gB(x, 2r_x) \cap B(x, 2r_x) = \emptyset$ or $g \in \mathrm{Stab}_G(x)$, and that the stabiliser $\mathrm{Stab}_G(x)$ is finite. Let $U^x := \bigcup\{gB(x, 2r_x) : g \in G\}$ and $V^x := \bigcup\{gB(x, r_x) : g \in G\}$. It is easy to verify that: assigning to an element gy, where $g \in G$ and $y \in B(x, 2r_x)$, the coset $g\mathrm{Stab}_G(x)$ gives a well-defined G-equivariant map $U^x \to G/\mathrm{Stab}_G(x)$; and that the map $\varphi^x\colon X \to [0, \infty)$ defined by $\varphi^x(y) = \sum_{g \in G} \max(0, r_x - d(gx, y))$ is a well-defined G-invariant continuous map that is non-zero precisely in the set V^x. Cocompactness of the action of G now gives that there exists a finite set $\{x_i : i \in I\}$ such that the sets V^{x_i} for $i \in I$ form an open cover of X; let $U_i := U^{x_i}$ and $\varphi_i := \varphi^{x_i}$ for $i \in I$. The collection $\{U_i : i \in I\}$ satisfies the property (a). It is easy to check that the functions $X \ni y \mapsto \varphi_i(y) / \sum_{j \in I} \varphi_j(y)$ for $i \in I$ form a locally finite partition of unity subordinate to $\{U_i : i \in I\}$, and are G-invariant since the functions φ_i are G-invariant. Property (ii) is satisfied by Proposition 1.3. Property (iii) is satisfied by the fact that the bicombing σ

gives the desired homotopy.

Observe that α is a morphism from (G, Σ_G) to (X, Σ), as for any $h \in G$ and $n \in \mathbb{N}$ we have that $d(\alpha(\Sigma_G(h,n)), \Sigma(\alpha(h),n)) = d(\alpha(\beta(\Sigma(\alpha(h),n))), \Sigma(\alpha(h),n))$ is universally bounded since $\alpha\beta$ is at bounded distance from the identity of X. Therefore, by [EW23, Lemma 2.8], in order to prove coherence and expandingness of Σ_G, it suffices to check coherence and expandingness of Σ.

The fact that Σ is coherent follows directly from consistency of σ (it suffices to take $R = 0$ in the definition of coherent combing).

Expandingness (cf. [EW23, Definition 2.6]) of Σ is equivalent to: there exists $R \geq 0$ such that for every $r \geq 0$ and $n \in \mathbb{N}$ there exists $D \geq 0$ such that $\Sigma_n(\overline{B}(x,r)) \subseteq \overline{B}(\Sigma_n(x), R)$ for all $x \in X \setminus \overline{B}(o, D)$. In fact, this statement holds for all $R > 0$, and this is what we will show. Fix any $R > 0$, $r \geq 0$, $n \in \mathbb{N}$, and take $x, x' \in X$ with $d(o,x) \geq n$ and $d(x,x') \leq r$. Then by Proposition 2.2 we have $d(\Sigma_n(x), \Sigma_n(x')) \leq 2nr/d(o,x)$, thus it suffices to take $D = \max(n, 2nr/R)$.

A combing Σ_Y on a space Y admitting an action of G is G-coarsely equivariant if the action of each $g \in G$ on Y induces an endomorphism of (Y, Σ_Y) (see [EW23, Definition 5.14]). To show that the combing Σ_G is G-coarsely equivariant, we first show that the combing Σ is G-coarsely equivariant, which means that for all $g \in G$ there exists $R(g) \geq 0$ such that $d(\Sigma_n(gx), g\Sigma_n(x)) \leq R(g)$ for all $x \in X$ and $n \in \mathbb{N}$. Indeed, take $g \in G$, $x \in X$ and $n \in \mathbb{N}$. Since σ is G-equivariant, $g\sigma_{ox} = \sigma_{go,gx}$. If $n \leq d(o,x)$ and $n \leq d(o,gx)$, we have that

$$\begin{aligned}
&d(\Sigma_n(gx), g\Sigma_n(x)) \\
&\leq d\left(\sigma_{o,gx}\left(\frac{n}{d(o,gx)}\right), \sigma_{o,gx}\left(\frac{n}{d(go,gx)}\right)\right) + d\left(\sigma_{o,gx}\left(\frac{n}{d(go,gx)}\right), \sigma_{go,gx}\left(\frac{n}{d(go,gx)}\right)\right) \\
&\qquad\qquad\qquad \leq \frac{n|d(go,gx) - d(o,gx)|}{d(o,gx)d(go,gx)} d(o,gx) + \left(1 - \frac{n}{d(go,gx)}\right) d(o,go) \\
&\qquad\qquad\qquad\qquad\qquad \leq d(o,go)\left(\frac{n}{d(go,gx)} + 1 - \frac{n}{d(go,gx)}\right) = d(o,go).
\end{aligned}$$

If $n \geq d(o,x)$ (equivalently, $\Sigma_n(x) = x$), then

$$\begin{aligned}
d(\Sigma_n(gx), g\Sigma_n(x)) &= d(\Sigma_n(gx), gx) = \max(0, d(o,gx) - n) \\
&\leq \max(0, d(o,gx) - d(o,x)) \leq |d(o,gx) - d(go,gx)| \leq d(o,go).
\end{aligned}$$

If $n \geq d(o,gx)$ (equivalently, $\Sigma_n(gx) = gx$), then, by the above with gx and g^{-1} in the place of x and g, respectively, we obtain $d(\Sigma_n(x), g^{-1}\Sigma_n(gx)) \leq d(o, g^{-1}o)$, which by g-invariance of the metric d gives $d(g\Sigma_n(x), \Sigma_n(gx)) \leq d(go, o)$. Summarising, it suffices to take $R(g) = d(o, go)$, and Σ is G-coarsely equivariant.

Now we show that Σ_G is G-coarsely equivariant. Let $g \in G$. By G-equivariance of α we have that $\alpha g = g\alpha$, therefore $\beta\alpha g = \beta g\alpha$. The right-hand side is a morphism as a composition of morphisms (β is a morphism by Proposition 3.1(i) as a quasi-inverse of α), see [EW23, Remark 2.18], therefore $\beta\alpha g$ is an endomorphism of (G, Σ_G). Since $\beta\alpha$ is at finite distance from the identity map of G, the map $\beta\alpha g$ is at finite distance from g; therefore, by Proposition 3.1(ii), the action of g on G induces an endomorphism of (G, Σ_G). Therefore Σ_G is G-coarsely equivariant. □

In the remaining part of this subsection we show that all EZ-structures resulting from the proof above (for the choice involved, see Remark 3.2-3 below) are equivalent to the EZ-structure resulting from the compactification $\overline{X}^{\Sigma}$ one would naturally consider (where the combing Σ is as above, and its definition is recalled in Remark 3.2-1), and which is among the EZ-structures resulting from the discussed proof (see Metaremark 3.3(b)). For a precise meaning of 'equivalent', see Proposition 3.4(ii).

We begin with a summary of the construction of the compactification from the proof above.

Remark 3.2. Formally, the EZ-structure constructed in the proof of Theorem I above is $(\overline{X}^{\Sigma'}, \overline{G}^{\Sigma_G} \setminus G)$ resulting from the following multistep procedure. (If the reader finds any of the steps below unsettling, they may refer to Metaremark 3.3 below.)

In the first paragraph of the proof above we perform the following steps.

1. Fix a basepoint $o \in X$. Define a combing Σ of X by $\Sigma(x, n) := \varrho_{o,x}(n)$.
2. Let α be a G-equivariant quasi-isometry given by $G \ni g \mapsto go \in X$ (the Švarc–Milnor Lemma) and $\beta\colon X \to G$ be its quasi-inverse. Define a combing Σ_G of G by $\Sigma_G(g, n) := \beta(\Sigma(\alpha(g), n))$.

Then we apply [EW23, Theorem 7.10 with Remark 7.15], which states that $(X^{\Sigma'}, G^{\Sigma_G} \setminus G)$ — which is the result of the next two steps — is an EZ-structure for G. (See [EW23, below Lemma 7.4, and proof of Theorem 7.10].)

3. Let $\alpha'\colon G \to X$ be a quasi-isometry of the form $\alpha'(g) = gx_0$ for some $x_0 \in X$, and let Σ' be a combing on X such that α' is a morphism from (G, Σ_G) to (X, Σ').
4. By functoriality of the construction of the boundary, [EW23, Corollary 2.18a], and Proposition 3.1, the morphism α' induces a homeomorphism from the boundary $\overline{G}^{\Sigma_G} \setminus G$ of G to the boundary $\overline{X}^{\Sigma'} \setminus X$ of X. Finally, the obtained EZ-structure for G is $(\overline{X}^{\Sigma'}, \overline{G}^{\Sigma_G} \setminus G)$, which is obtained from $(\overline{X}^{\Sigma'}, \overline{X}^{\Sigma'} \setminus X)$ using this identification of boundaries.

In view of the identification of $(\overline{X}^{\Sigma'}, \overline{G}^{\Sigma_G} \setminus G)$ with $(\overline{X}^{\Sigma'}, \overline{X}^{\Sigma'} \setminus X)$, we prefer to use the latter instead of the former, and consider in the text below $\overline{X}^{\Sigma'}$ to be the compactification resulting from the proof of Theorem I from this subsection.

Metaremark 3.3. (a) One may ask the question why we do not finish the construction at Step 1, and simply equip the space X with the combing Σ. The (formal) reason is that in the proof of Theorem I in this subsection we used [EW23, Theorem 7.10] as a black box, in which it is the group G that is the object equipped with the combing, while the space X is required to be G-equivariantly homotopy equivalent with $\mathrm{Rips}(G)$ and to be an ER. This is an example of a difference between the approaches in this paper and in [EW23]: while the natural place for (bi)combings in this paper are topological spaces, Engel and Wulff tend to prefer to have the combing on the group G or its Rips complex $\mathrm{Rips}(G)$ — while the latter has nice abstract properties, and for it they can perform the homotopy from the definition of the Z-set, [EW23, Theorem 5.7], it does not have nice topological properties. This is the reason why they construct the compactification using $\mathrm{Rips}(G)$ and exchange the space that has been compactified for X. (See [EW23, beginning of Section 7]).

(b) We note that the choice $\alpha' := \alpha$ and $\Sigma' := \Sigma$ satisfies the properties required in Remark 3.2-3 (as has been shown in the proof of Theorem I in this subsection), therefore $\overline{X}^{\Sigma}$ is one of the compactifications constructed in the proof of Theorem I in this subsection.

Proposition 3.4. *Let G be a group acting geometrically on a proper metric space X that admits a $\mathfrak{ccc}$, G-equivariant bicombing σ, and let Σ, Σ' be as in Remark 3.2. Then:*

(i) the identity function $\mathrm{id}_X \colon X \to X$ is a morphism from (X, Σ) to (X, Σ');

(ii) the function $\mathrm{id}_X \colon X \to X$ induces a G-equivariant homeomorphism $(\mathrm{id}_X)_* \colon \overline{X}^{\Sigma} \to \overline{X}^{\Sigma'}$ fixing the copies of X in $\overline{X}^{\Sigma}$ and $\overline{X}^{\Sigma'}$ pointwise, that is $(\mathrm{id}_X)_*(\delta_x) = \delta_x$ for all $x \in X$.

Remark 3.5. (i) (cf. [EW23, Remark 7.15]) Regarding the action of G on the compactifications mentioned in the statement (ii) above, we justify below that the action of each $g \in G$ on X induces endomorphisms of (X, Σ) and (X, Σ'); these induce by functoriality, [EW23, Corollary 2.18b], the desired homeomorphisms $g_* \colon \overline{X}^{\Sigma} \to \overline{X}^{\Sigma}$ and $g_* \colon \overline{X}^{\Sigma'} \to \overline{X}^{\Sigma'}$. The fact that the action of g induces an endomorphism of (X, Σ) has been proved previously in this section in the proof of Theorem I. In order to justify that g induces an endomorphism of (X, Σ'), observe that by construction of α' we have that $g\alpha' = \alpha' g$, therefore $g\alpha'\beta = \alpha' g\beta$. The right-hand side is a morphism as a composition of 3 morphisms (β is a morphism from (X, Σ') to (G, Σ_G) by Proposition 3.1(i) as a quasi-inverse of α'). The left-hand side is at finite distance from g, as g acts on X via isometries and $\alpha'\beta$ is at finite distance from the identity of X. Therefore, by Proposition 3.1(ii), the action of g on X induces an endomorphism of (X, Σ').

(ii) The functoriality of the construction of the compactification in [EW23, Corollary 2.18] works in the following way. It is a standard fact that a continuous function F induces a pullback map F^* between the spaces of continuous functions, and a pushforward map F_* between the spaces dual to these spaces of continuous functions. It turns out that if F is a morphism of combed spaces, then F_* gives a well-defined continuous map between the compactifications.

Proof. (i) Since the quasi-isometry α' is a morphism from (G, Σ_G) to (X, Σ'), and β is a morphism from (X, Σ) to (G, Σ_G) (by Proposition 3.1(i), as a quasi-inverse of α, which we have shown to be a morphism from (G, Σ_G) to (X, Σ) in the proof of Theorem I in this subsection), the map $\alpha'\beta$ is a morphism from (X, Σ) to (X, Σ'). Since both functions α' and α are of the form $g \mapsto gx$ for some $x \in X$, they are at finite distance from each other, so the function $\alpha'\beta$ is at finite distance from $\alpha\beta$, which is at finite distance from the identity $\mathrm{id_X}$ of X, which implies by Proposition 3.1(ii) that $\mathrm{id_X}$ is a morphism from (X, Σ) to (X, Σ').

(ii) By Proposition 3.1(i), the identity id_X is also a morphism from (X, Σ') to (X, Σ), therefore, by functoriality, [EW23, Corollary 2.18b], the induced function $(\mathrm{id}_X)_*\colon \overline{X}^{\Sigma} \to \overline{X}^{\Sigma'}$ is a homeomorphism. Finally, observe that for all $x \in X$ we have that $(\mathrm{id}_X)_*\delta_x = \delta_x$, and for all $g \in G$ we have that $g\mathrm{id}_X = \mathrm{id}_X g$, which implies that $g_*(\mathrm{id}_X)_* = (\mathrm{id}_X)_* g_*$. $\square$

3.2. Equivalence of the constructed compactifications

In this section we prove that the constructions of the EZ-structures from Subsections 2.1 and 3.1 produce the same compactification.

We note here that the pre-'moreover' part of the proposition below follows from a stronger result, holding in a more general setting, [FO20, Corollary 8.9], in a way described in [EW23, above Examples 3.27, and 3.27.2]. In view of this, we may see the proof of the proposition below as a more elementary proof of a special case (with the minor addition of G-equivariance to the considerations).

Proposition 3.6. *Let (X, d) be a proper metric space that admits a* ccc *bicombing σ. Let Σ be as in Remark 3.2-1. Then there exists a homeomorphism $\tau\colon \overline{X}_\sigma \to \overline{X}^{\Sigma}$ fixing the copies of X in $\overline{X}_\sigma$ and $\overline{X}^{\Sigma}$ pointwise, i.e. $\tau(x) = \delta_x$ for all $x \in X$. Moreover, if a group*

G acts on X in such a way that the bicombing σ is additionally G-equivariant, then τ is additionally G-equivariant.

In view of Proposition 3.4 and Remark 3.2, we have the following corollary.

Corollary 3.7. *Let G be a group acting geometrically on a proper metric space X that admits a $\mathfrak{ccc}$, G-equivariant bicombing σ, and let $\overline{X}_\sigma$, $\overline{X}^{\Sigma'}$ be the compactifications of X constructed in the proofs of Theorem I in Subsections 2.1, 3.1, respectively. Then there exists a G-equivariant homeomorphism $\tau\colon \overline{X}_\sigma \to \overline{X}^{\Sigma'}$ fixing the copies of X in $\overline{X}_\sigma$ and $\overline{X}^{\Sigma'}$ pointwise, i.e. $\tau(x) = \delta_x$ for all $x \in X$.*

Proof. (of Proposition 3.6) Put $\tau(\bar{x}) := \delta_{\bar{x}}$, where $\delta_{\bar{x}}(f) = \lim_n f(\varrho_{o,\bar{x}}(n))$ for all $f \in C_\Sigma(X)$.

Observe that for any $x \in X$ and $n \geq d(o,x)$ we have that $\varrho_{o,x}(n) = x$, therefore the definition of δ from the definition of τ extends the previous definition of δ from Section 3 (from X to the whole $\overline{X}_\sigma$), in particular the equality $\tau(x) = \delta_x$ holds for all $x \in X$. Observe that for any $\bar{x} \in \overline{X}_\sigma$ the limit $\lim_n f(\varrho_{o,\bar{x}}(n))$ exists, as the convergence of $f \circ \Sigma_n$ to f in the supremum metric implies that the diameters of the sets $(f \circ \varrho_{o,\bar{x}})([n,\infty))$ converge to 0 when $n \to \infty$, and that $\delta_{\bar{x}}$ is a non-zero multiplicative linear functional on $C_\Sigma(X)$; therefore the map τ is well-defined.

The map τ is one-to-one by the following argument. Let $\bar{x} \in \overline{X}_\sigma$. Define $\mathfrak{d}_{\bar{x}}(x) := d_o(\bar{x}, x)$ (the metric d_o has been introduced in equation (2-1)). The function $\mathfrak{d}_{\bar{x}}$ is bounded (by 1) and continuous on X. Furthermore, for all $x \in X$ and $n \in \mathbb{N}$ we have that $|\mathfrak{d}_{\bar{x}}(\Sigma_n(x)) - \mathfrak{d}_{\bar{x}}(x)| \leq 2^{-n+1}$, as for all $i \leq n$ we have by consistency of σ that $\varrho_{o,\Sigma_n(x)}(i) = \varrho_{o,\varrho_{o,x}(n)}(i) = \varrho_{o,x}(i)$, so the i-th summands for $i \leq n$ in the definitions of $d_o(\bar{x}, x)$ and $d_o(\bar{x}, \Sigma_n(x))$ are the same; this implies that $\mathfrak{d}_{\bar{x}} \circ \Sigma_n$ converges to $\mathfrak{d}_{\bar{x}}$ in the supremum metric. To see that $\mathfrak{d}_{\bar{x}}$ has bounded variation, take $R > 0$, $x \in X$ and $y \in \overline{B}(x, R)$, and let $N \in \mathbb{N}$ be the integer part of $d(o,x)$. By the triangle inequality and Proposition 2.2 we have that

$$
\begin{aligned}
|\mathfrak{d}_{\bar{x}}(x) - \mathfrak{d}_{\bar{x}}(y)| \leq \sum_{n=1}^{N} 2^{-n} d(\varrho_{o,x}(n), \varrho_{o,y}(n)) + \sum_{n=N+1}^{\infty} 2^{-n} \\
\leq \sum_{n=1}^{\infty} \frac{2Rn}{d(o,x)} 2^{-n} + 2^{-N} \leq \frac{4R}{d(o,x)} + 2^{-d(o,x)+1},
\end{aligned}
$$

which tends to 0 when $d(o,x)$ tends to ∞. Therefore $\mathfrak{d}_{\bar{x}} \in C_\Sigma(X)$. By continuity of d_o, for any $\bar{y} \in \overline{X}_\sigma$ we have that $\tau(\bar{y})(\mathfrak{d}_{\bar{x}}) = \delta_{\bar{y}}(\mathfrak{d}_{\bar{x}}) = d_o(\bar{y}, \bar{x})$. Therefore, if $\tau(\bar{y}) = \tau(\bar{x})$, then by an application of both sides to $\mathfrak{d}_{\bar{x}}$ we obtain that $d_o(\bar{y}, \bar{x}) = d_o(\bar{x}, \bar{x})$, which is equal to 0, so $\bar{x} = \bar{y}$.

The map τ is onto $\overline{X}^{\Sigma}$ by the following argument. Assume that there exists a maximal proper ideal I of $C_\Sigma(X)$ such that $I \neq \ker \delta_{\bar{x}}$ for any $\bar{x} \in \overline{X}_\sigma$. Then, because I is closed under the multiplication by the elements of the (whole) algebra $C_\Sigma(X)$, we can pick a family of functions $\{f_{\bar{x}} : \bar{x} \in \overline{X}_\sigma\} \subseteq I$ such that $f_{\bar{x}}$ is non-negative and $\delta_{\bar{x}}(f_{\bar{x}}) = 1$. If $x \in X$, then there exists $r_x > 0$ such that $f_{\bar{x}} \geq 1/2$ on $B(x, r_x)$. If $\bar{x} \in \partial_\sigma X$, then, since $f_{\bar{x}} \in C_\Sigma(X)$, we can choose $t_{\bar{x}} \in \mathbb{N}$ such that for all $x \in X$ with $d(o, x) \geq t_{\bar{x}}$ we have that $|f_{\bar{x}}(x) - f_{\bar{x}}(\Sigma_{t_{\bar{x}}}(x))| \leq 1/6$ (as $f_{\bar{x}} \circ \Sigma_n \to f_{\bar{x}}$) and $|f_{\bar{x}}(x) - f_{\bar{x}}(y)| \leq 1/6$ for all $y \in X$ with $d(x, y) \leq 1$ (as $f_{\bar{x}}$ has bounded variation). Then we have by the triangle inequality that for all $y \in U_o(\bar{x}, t_{\bar{x}}, 1) \cap X$ and $n \geq t_x$

$$\begin{aligned}|f_{\bar{x}}(y) - f_{\bar{x}}(\varrho_{o,\bar{x}}(n))| &\leq |f_{\bar{x}}(y) - f_{\bar{x}}(\varrho_{o,y}(t_{\bar{x}}))| + |f_{\bar{x}}(\varrho_{o,y}(t_{\bar{x}})) - f_{\bar{x}}(\varrho_{o,\bar{x}}(t_{\bar{x}}))| \\ &\quad + |f_{\bar{x}}(\varrho_{o,\bar{x}}(t_{\bar{x}})) - f_{\bar{x}}(\varrho_{o,\bar{x}}(n))| \leq 1/6 + 1/6 + 1/6 = 1/2,\end{aligned}$$

therefore, passing to the limit with n, we obtain that $f_{\bar{x}}(y) \geq 1 - 1/2 = 1/2$. Observe that the family $\{U_o(\bar{x}, t_{\bar{x}}, 1) : \bar{x} \in \partial_\sigma X\} \cup \{B(x, r_x) : x \in X\}$ is an open cover of $\overline{X}_\sigma$, thus we can choose a finite subcover $\{U_o(\bar{x}_i, t_{\bar{x}_i}, 1) : i = 1, \ldots, m_1\} \cup \{B(x_j, r_{x_j}) : j = 1, \ldots, m_2\}$. Then the function $f := f_{\bar{x}_1} + \ldots + f_{\bar{x}_{m_1}} + f_{x_1} + \ldots + f_{x_{m_2}}$ belongs to I and $1/2 \leq f(x)$ for all $x \in X$. Using the fact that

$$\left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| = \frac{|f(x) - f(y)|}{|f(x)f(y)|} \leq 4|f(x) - f(y)|$$

for all $x, y \in X$, one may verify that the function $1/f$ belongs to $C_\Sigma(X)$, therefore $1 = (1/f) \cdot f \in I$ and $I = C_\Sigma(X)$. This contradicts properness of I.

Since $\overline{X}_\sigma$ is compact and τ is bijective, in order to prove that τ is a homeomophism, it suffices to show that it is continuous. Consider any subbase open subset $U_{f,a,\epsilon} := \{\delta_{\bar{y}} \in \overline{X}^{\Sigma} : |\delta_{\bar{y}}(f) - a| < \epsilon\}$ of $\overline{X}^{\Sigma}$, where $f \in C_\Sigma(X)$, $a \in \mathbb{C}$ and $\epsilon > 0$. Then $\tau^{-1}(U_{f,a,\epsilon}) = \{\bar{y} \in \overline{X}_\sigma : |\delta_{\bar{y}}(f) - a| < \epsilon\}$. Consider a point $\bar{x} \in \overline{X}_\sigma$ such that $\epsilon' := |\delta_{\bar{x}}(f) - a| < \epsilon$. If $\bar{x} \in X$, then by continuity of f there exists $\eta > 0$ such that

$$\{\delta_y(f) : y \in B(\bar{x}, \eta)\} = f(B(\bar{x}, \eta)) \subseteq B_{\mathbb{C}}(\delta_{\bar{x}}(f), \epsilon - \epsilon') \subseteq B_{\mathbb{C}}(a, \epsilon)$$

If $\bar{x} \in \partial_\sigma X$, then, as in the proof that τ is onto, by the fact that $f \circ \Sigma_n \to f$ and that f has bounded variation, for large enough t we have that

$$f(U_o(\bar{x}, t, 1) \cap X) \subseteq \overline{B}_{\mathbb{C}}(\delta_{\bar{x}}(f), (\epsilon - \epsilon')/2) \subseteq \overline{B}_{\mathbb{C}}(a, \epsilon' + (\epsilon - \epsilon')/2),$$

therefore $\{\delta_{\bar{y}}(f) : \bar{y} \in U_o(\bar{x}, t, 1)\} \subseteq \overline{f(U_o(\bar{x}, t, 1) \cap X)} \subseteq B_{\mathbb{C}}(a, \epsilon)$.

Now we prove the 'moreover' part. Observe that for any $g \in G$ and $x \in X$ we have $g_*\tau(x) = g_*\delta_x = \delta_{gx} = \tau(gx)$. Since τ is continuous, and the actions of G on $\overline{X}_\sigma$ and $\overline{X}^{\Sigma}$ are via continuous functions, and X is dense in $\overline{X}_\sigma$, we have that $g_*\tau(\bar{x}) = \tau(g\bar{x})$ for any $\bar{x} \in \overline{X}_\sigma$. $\square$

4. Non-uniqueness of boundary

In this section we adapt the classical example by Croke and Kleiner [CK00] to prove the following theorem on non-uniqueness of the boundary defined in Section 2 in the case of injective groups. (For the definition of injective metric space, see the paragraph above Corollary II.)

Theorem 4.1 (Theorem IV). *There exists a group G acting geometrically on two proper finite-dimensional injective metric spaces $X^{\diamondsuit}, X^{\square}$ with convex bicombings $\sigma^{\diamondsuit}, \sigma^{\square}$, respectively, such that $\partial_{\sigma^{\diamondsuit}} X^{\diamondsuit}$ and $\partial_{\sigma^{\square}} X^{\square}$ are not homeomorphic.*

Remark 4.2. As it has been discussed in Remark III and in the proof of Corollary II(i), a proper injective metric space X that is finite-dimensional — or, more generally, such that each of its bounded subsets is of finite dimension — admits a unique convex bicombing, which is additionally consistent, reversible and equivariant with respect to the full isometry group $\operatorname{Iso}(X)$ of X.

In a CAT(0) space X, each pair of points is connected by a unique geodesic; these geodesics give the unique bicombing on X, see [BH99, Proposition II.1.1.4(1)]. This bicombing therefore is automatically consistent, reversible and $\operatorname{Iso}(X)$-equivariant. It is also convex, see [BH99, Proposition II.2.2.2].

Further in this section, we will use these uniqueness results without mentioning.

We say that two bicombings $\sigma^{\circ}, \sigma^{\bullet}$ on a complete geodesic metric space X *have the same trajectories* if $\operatorname{im} \sigma^{\circ}_{xx'} = \operatorname{im} \sigma^{\bullet}_{xx'}$ for all $x, x' \in X$.

The strategy of the proof of Theorem 4.1 is as follows. We take a group acting on two spaces with non-homeomorphic boundaries, described in [CK00] (namely, the group of deck transformations acting on the universal covers of the spaces in Figure 2) and replace the original, piecewise-ℓ^2 metrics on these spaces with piecewise-ℓ^∞ metrics. It is then sufficient, see Proposition 4.8, to show that the resulting spaces are injective and equipped with bicombings having the same trajectories as the original CAT(0) ones. It turns out that these properties may be checked locally, to which end Lemma 4.7 serves. Lemmas 4.3, 4.4, 4.5 and Proposition 4.6 are some preparatory lemmas useful in the proof of Lemma 4.7.

We begin with the following simple, yet very useful observation.

Lemma 4.3. *Let φ be an isometry of a metric space X that possesses a φ-equivariant bicombing σ. Then the fixpoint set $\operatorname{Fix}(\varphi)$ of φ is σ-convex.*

Proof. For any $x, y \in \operatorname{Fix}(\varphi)$ we have that $\sigma_{xy} = \sigma_{\varphi(x)\varphi(y)} = \varphi \circ \sigma_{xy}$. □

We add some simple observations to the work of Miesch on gluings of injective metric spaces. Following [Mie15], we call a subset A of a metric space (X,d) *strongly convex* whenever for all $x,y\in A$ the metric interval $\{z\in X : d(x,y)=d(x,z)+d(z,y)\}$ is contained in A, and *externally hyperconvex* (recall the discussion about names from around the definition of injective metric space, above Corollary II) whenever for every family of points $x_i\in X$ and radii $r_i>0$ that satisfies $d(x_i,x_j)\le r_i+r_j$ and $d(x_i,A)\le r_i$, the set $A\cap\bigcap\overline{B}(x_i,r_i)$ is non-empty. For $\mathbb{R}^2$ with the ℓ^∞-metric, a standard example of a strongly convex subset is the diagonal line $\{(d,d) : d\in\mathbb{R}\}$, and of an externally hyperconvex subset is the horizontal line $\{(x,0) : x\in\mathbb{R}\}$.

Further in this chapter, by an *std-gluing* of the spaces $(X_\lambda,d_\lambda)_{\lambda\in\Lambda}$ along a subspace A we mean the metric space X obtained in the following process. We assume that we have a family of metric spaces $(X_\lambda,d_\lambda)_{\lambda\in\Lambda}$, a space (A,d_A), and isometric embeddings $\imath_\lambda\colon (A,d_A)\to(X_\lambda,d_\lambda)$ for all $\lambda\in\Lambda$, such that $\imath_\lambda(A)$ is closed in X_λ. Then X arises as the image of the gluing map π defined as the quotient map of the relation $\sim$ on $\bigsqcup_{\lambda\in\Lambda}X_\lambda$ given by $\imath_\lambda(a)\sim\imath_{\lambda'}(a)$ for all $\lambda,\lambda'\in\Lambda$ and $a\in A$, with the standard gluing metric. Note that the spaces A and X_λ can be isometrically embedded into X using π (and the $\imath_\lambda$), and therefore we will identify these spaces with their images in X.

In the following lemma, the statements about CAT(0)-ness and about injectivity, see [Mie15], of the spaces resulting from gluings have been known previously.

Lemma 4.4. *Let X be the std-gluing of metric spaces $(X_\lambda,d_\lambda)_{\lambda\in\Lambda}$ along some space A. In any of the 3 cases below:*

1. *the spaces X_λ are CAT(0), and A is a closed and convex subset of each of the X_λ;*
2. *the spaces X_λ are injective, X is proper and finite-dimensional, and A is externally hyperconvex (therefore, automatically, closed) in each of the X_λ;*
3. *the spaces X_λ are injective, X is proper and finite-dimensional, and A is closed and strongly convex in each of the X_λ;*

the space X is CAT(0) (case 1) or injective (cases 2 and 3), thus admits a convex bicombing σ; and for all $\Lambda_0\subseteq\Lambda$ the space $X_0:=A\cup\bigcup_{\lambda\in\Lambda_0}X_\lambda$ is σ-convex, in particular $\sigma|_{X_0\times X_0\times[0,1]}$ is the convex bicombing on X_0.

Proof. For each $\lambda\in\Lambda$ consider copies $X_\lambda^1,X_\lambda^2,X_\lambda^3$ of the space X_λ, and let the space $\widetilde{X}$ be the std-gluing of the family $\{X_\lambda^i : \lambda\in\Lambda, 1\le i\le 3\}$ along A. Note that the space X is a subspace of $\widetilde{X}$, and the latter consists of 3 copies of X std-glued along A. Applying [BH99, Theorem II.11.3], [Mie15, Theorem 1.3 and Theorem 1.1] to the cases 1, 2 and 3, respectively, we get that the spaces X_0, X, $\widetilde{X}$ are all CAT(0), or proper, finite-dimensional (see e.g. [Eng78, Theorem 3.1.4 and Proposition 3.1.7]) and injective, thus admit convex bicombings; denote by $\widetilde{\sigma}$ the convex bicombing on $\widetilde{X}$. Consider the action of the group $\mathrm{Sym}(\{1,2,3\})^{\Lambda}$ with the λ-th coordinate acting by permuting copies X_λ^j of the spaces X_λ. Observe that X is the set of fixed points of $((1)(2\,3))_{\lambda\in\Lambda}$, therefore by Lemma 4.3 it is $\widetilde{\sigma}$-convex and $\sigma := \widetilde{\sigma}|_{X\times X\times[0,1]}$ is the convex bicombing on X. Similarly, one can see that there exists an element of $\mathrm{Sym}(\{1,2,3\})^{\Lambda}$ with fixed point set equal to X_0, therefore by Lemma 4.3 the space X_0 is $\widetilde{\sigma}$-convex, thus σ-convex, and $\widetilde{\sigma}|_{X_0\times X_0\times[0,1]} = \sigma|_{X_0\times X_0\times[0,1]}$ is the convex bicombing on X_0. □

Lemma 4.5. *Let X be the std-gluing of injective spaces $(X_\lambda, d_\lambda)_{\lambda\in\Lambda}$ along some space A.*

(i) Assume that A is externally hyperconvex in each of the X_λ. Then for all subsets $\Lambda_0 \subseteq \Lambda$ the set $X_0 := A \cup \bigcup_{\lambda\in\Lambda_0} X_\lambda$ is an externally hyperconvex subset of X.

(ii) Let A be closed in each of the X_λ. Suppose $B \subseteq X_{\lambda_0}$ for some $\lambda_0 \in \Lambda$ is strongly convex in X_{λ_0} and $|A \cap B| \leq 1$. Then B is strongly convex in X.

Proof. (i) Let $\{\overline{B}(x_i, r_i)\}_{i\in I}$ be a collection of closed balls in X with $d(x_i, x_j) \leq r_i + r_j$ and $d(x_i, X_0) \leq r_i$. If this collection satisfies $d(x_i, A) \leq r_i$ for all $i \in I$, then the claim follows by the fact that A is externally hyperconvex in X by [Mie15, Theorem 1.3]. Otherwise, there exist $i_0 \in I$ and $\lambda_0 \in \Lambda_0$ such that $\overline{B}(x_{i_0}, r_{i_0}) \subseteq X_{\lambda_0}$. By [Mie15, Theorem 1.3], the space X is injective, therefore $\emptyset \neq \bigcap_{\lambda\in\Lambda_0} \overline{B}(x_i, r_i) \subseteq \overline{B}(x_{i_0}, r_{i_0}) \subseteq X_{\lambda_0} \subseteq X_0$. The claim follows.

(ii) Take a geodesic γ in X with endpoints in B. Since A is closed in each of the X_λ, the set X_{λ_0} is closed in X, therefore the preimage $\gamma^{-1}(X \setminus X_{\lambda_0})$ is open, thus it is a union of disjoint open intervals. Replace γ on each of such intervals with a geodesic contained in X_{λ_0}, obtaining a new geodesic γ' with the same endpoints as γ, which is contained in X_{λ_0}, and thus in B, by strong convexity. This implies that all of the endpoints of the intervals where γ went out of X_{λ_0} are contained in B. Since $|A \cap B| \leq 1$, it follows that $\gamma = \gamma'$, so γ is contained in B. The claim follows. □

We note that the following can be derived from [Mie17], where the proof is more involved, as the setting is more general.

Proposition 4.6. *Let X be a metric space that admits a* ccc *bicombing σ. Assume that $c\colon [0,1] \to X$ is a constant speed geodesic such that locally it is a σ-geodesic, i.e. there exists an open cover $\mathcal{U}$ of $[0,1]$ such that $\operatorname{im}\sigma_{c(s)c(t)} = \operatorname{im} c|_{[s,t]}$ for all $U \in \mathcal{U}$ and $s, t \in U$ satisfying $s \leq t$. Then c is the σ-geodesic $\sigma_{c(0)c(1)}$.*

Proof. Consider the following statement $\mathtt{S}(l)$, where $l \in [0,1]$: for all $s, t \in [0,1]$ such that $s \leq t$ and $t - s \leq l$ the equality $\operatorname{im}\sigma_{c(s)c(t)} = \operatorname{im} c|_{[s,t]}$ holds. By compactness of $[0,1]$, the statement $\mathtt{S}(\epsilon)$ holds for a sufficiently small $\epsilon > 0$. It is sufficient to show that for any $l \in [0,1]$ the statement $\mathtt{S}(2l/3)$ implies $\mathtt{S}(l)$. Take any $s,t \in [0,1]$ such that $s \leq t$ and $t-s \leq l$. Let $p := c((2s+t)/3)$, $q := c((s+2t)/3)$ and $p' := \sigma_{c(s)c(t)}(1/3)$, $q' := \sigma_{c(s)c(t)}(2/3)$. By conicality and $\mathtt{S}(2l/3)$, we have $d(p,p') \leq d(q,q')/2$. Similarly, $d(q,q') \leq d(p,p')/2$. Therefore $d(p,p') = 0 = d(q,q')$, and, by consistency of σ, $\operatorname{im} \sigma_{c(s)c(t)} = \operatorname{im} c|_{[s,t]}$. $\square$

For a space X obtained by gluing up to 3 copies $P_j := \{(x,y)^{P_j} : x,y \in \mathbb{R}\}$ of $\mathbb{R}^2$, where $j = 1,2,3$, we denote by X^∞ (resp. X^2) the space X equipped with the metric arising from putting the ℓ^∞-metric (resp. the ℓ^2-metric) on each of the P_j.

The main technical lemma we use to adapt the Croke–Kleiner example is as follows.

Lemma 4.7. *Let X be one of the following spaces:*

1. $\mathbb{R}^2$,
2. $P_1 \sqcup P_2 \,/\{(x,0)^{P_1} = (x,0)^{P_2} : x \in \mathbb{R}\}$,
3. $P_1 \sqcup P_2 \,/\{(d,d)^{P_1} = (d,d)^{P_2} : d \in \mathbb{R}\}$,
4. $P_1 \sqcup P_2 \sqcup P_3 \,/\{(x,0)^{P_1} = (x,0)^{P_2} : x \in \mathbb{R}\} \cup \{(0,y)^{P_2} = (0,y)^{P_3} : y \in \mathbb{R}\}$,
5. $P_1 \sqcup P_2 \sqcup P_3 \,/\{(x,0)^{P_1} = (x,0)^{P_2} : x \in \mathbb{R}\} \cup \{(d,d)^{P_2} = (d,d)^{P_3} : d \in \mathbb{R}\}$.

Then X^2 is CAT(0), X^∞ is injective, and the convex bicombings σ^2 of X^2 and σ^∞ of X^∞ have the same trajectories.

Proof. The space X^2 is CAT(0) in each of the cases by [BH99, Remark II.11.2.2].

Let X be as in 1. It is well-known that the space X^∞ is injective. Both σ^∞ and σ^2 consist of linear segments — one can refer to [DL15, Theorem 3.3].

Let X be as in 2. The resulting space X^∞ is injective by Lemma 4.4, as $\{(x,0) : x \in \mathbb{R}\}$ is an externally hyperconvex subset of $\mathbb{R}^2$ with the ℓ^∞-metric. Furthermore, for any $p \in \{2,\infty\}$ the space X^p restricted to the union of any pair of the 4 closed halfplanes induced in P_1 and P_2 by the x-axis (i.e. $\{(x,y)^{P_j} : yR0, x \in \mathbb{R}\}$, where $R \in \{\leq,\geq\}$ and $j = 1,2$) is isometric to $\mathbb{R}^2$ with the ℓ^p-metric; therefore, by Lemma 4.4 and case 1, we obtain that $\sigma^2 = \sigma^\infty$.

Let X be as in 3. The argument is analogous to the one in 2, with the difference that the space X^∞ is injective as the set $\{(d,d) : d \in \mathbb{R}\}$ is strongly convex in $\mathbb{R}^2$ with the ℓ^∞-metric.

Let X be as in 4. The space X may be seen as the std-gluing of the spaces $X_{1,2}$ and $X_{2,3}$ along P_2, where the space $X_{1,2}$ is the std-gluing of P_1 with P_2 along the x-axis and $X_{2,3}$ is the std-gluing of P_2 with P_3 along the y-axis. The set P_2 is externally hyperconvex in both $X^\infty_{1,2}$ and $X^\infty_{2,3}$ by Lemma 4.5(i); therefore, by Lemma 4.4, the space X^∞ is injective,

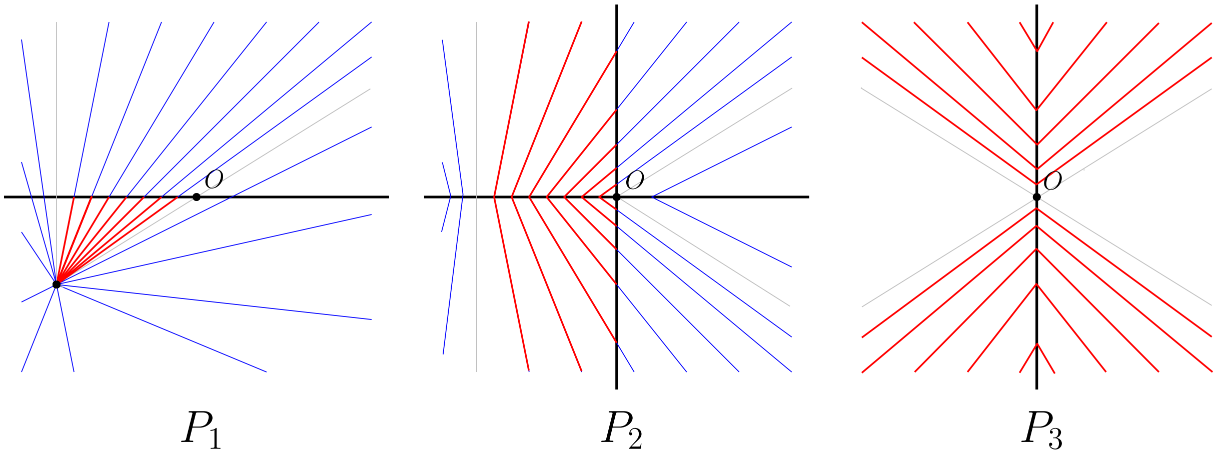

Figure 1: Paths, whose intersections with $P_1 \cup P_2$ and $P_2 \cup P_3$ are linear segments, originating in some point $a \in P_1 \setminus (P_2 \cup P_3)$ and omitting O (the red ones are these that reach P_3).

and the subsets $P_1 \cup P_2$ and $P_2 \cup P_3$ are both σ^∞-convex and σ^2-convex in X. Therefore, since case 2 applies to both $X_{1,2}$ and $X_{2,3}$, it remains to show that for all $a \in P_1 \setminus (P_2 \cup P_3)$ and $b \in P_3 \setminus (P_1 \cup P_2)$ we have $\operatorname{im} \sigma^\infty_{ab} = \operatorname{im} \sigma^2_{ab}$. Let $O := (0,0)^{P_1} (= (0,0)^{P_2} = (0,0)^{P_3})$ and $\sigma \in \{\sigma^\infty, \sigma^2\}$. By case 2 (applied to $X_{1,2}$ and to $X_{2,3}$), the sets $\operatorname{im} \sigma_{ab} \cap (P_1 \cup P_2)$ and $\operatorname{im} \sigma_{ab} \cap (P_2 \cup P_3)$ are linear segments (when considered as subsets of appropriate planes in $X_{1,2}$ or $X_{2,3}$, see Figure 1). If there exists a path ϖ from a to b such that ϖ omits O, and $\operatorname{im} \varpi \cap (P_1 \cup P_2)$ and $\operatorname{im} \varpi \cap (P_2 \cup P_3)$ are linear segments, then, after an appropriate reparametrisation, ϖ is locally a σ-geodesic, therefore by Proposition 4.6 is the σ-geodesic σ_{ab}. If there is no such path from a to b, then the image $\operatorname{im} \sigma_{ab}$ contains O, and therefore is equal to the chain of segments with vertices a, O, b. The case follows.

Let X be as in 5. By Lemma 4.5(ii), the subset $\{(d,d)^{P_2} : d \in \mathbb{R}\}$ is strongly convex in the std-gluing of P_1 with P_2 along the x-axis; therefore, by Lemma 4.4, the space X^∞ is injective, and the subset $P_1 \cup P_2$ is σ^2-convex and σ^∞-convex in X. The subset $P_2 \cup P_3$ is σ^2-convex and σ^∞-convex in X as a consequence of Lemma 4.3 applied to the map $X \to X$ fixing $P_2 \cup P_3$ pointwise and reflecting P_1 with respect to its x-axis. The remaining part of the argument is analogous to the one in 4. (Side note: despite the similarities in the proofs, the analogue of Figure 1 for case 5 is substantially different from Figure 1, e.g. it is not just a sheared version of it.) $\square$

Proposition 4.8. *Let $d_\circ$ and $d_\bullet$ be two complete metrics on a topological space X, and assume that for $i \in \{\circ, \bullet\}$ the space (X, d_i) admits a* ccc *bicombing σ^i. Assume that σ° and $\sigma^\bullet$ have the same trajectories. Then the identity id_X continuously extends to a homeomorphism $\iota\colon \overline{X}_{\sigma^\circ} \to \overline{X}_{\sigma^\bullet}$.*

Proof. Consider a σ°-ray $\xi_\circ$. Let $l_\bullet(t) := d_\bullet(\xi_\circ(0), \xi_\circ(t))$. Since the bicombings σ° and $\sigma^\bullet$ have the same trajectories, the function $l_\bullet(t)$ is increasing. We show that $l_\bullet(t)$ is unbounded. If it was bounded, then it would have some limit. Therefore, $(\xi_\circ(n))_{n\in\mathbb{N}}$ would be a Cauchy sequence in $(X, d_\bullet)$, so by completeness of the metric $d_\bullet$ on the space X, it would have a limit in X. On the other hand, $\{\xi_\circ(n) : n \in \mathbb{N}\}$ is a discrete set in $(X, d_\circ)$, therefore the sequence $(\xi_\circ(n))_{n\in\mathbb{N}}$ is not convergent in X — a contradiction. Note that the argumentation contained in the previous part of this paragraph works with $\circ$ and $\bullet$ swapped.

Fix a basepoint $o \in X$. Let the map $\iota\colon \overline{X}_{\sigma^\circ} \to \overline{X}_{\sigma^\bullet}$ be induced by the map that is the identity on X and assigns to the σ°-ray $\varrho^\circ_{o,\bar{x}}$ the $\sigma^\bullet$-ray originating in o with the same image as $\varrho^\circ_{o,\bar{x}}$. By the previous paragraph, the map ι is well-defined and restricts to a bijection from the boundary $\partial_{\sigma^\circ}X$ to the boundary $\partial_{\sigma^\bullet}X$.

Now we show continuity of ι. Clearly, ι is continuous at every point of X, as X is an open subset of $\overline{X}_{\sigma^\circ}$ and ι restricted to X is the identity. Consider any $\bar{x} \in \partial_{\sigma^\circ}X$ and $t_\bullet, \epsilon_\bullet > 0$. Let $t_\circ := d_\circ(\varrho^\bullet_{o,\iota(\bar{x})}(t_\bullet))$ (so that, in particular, $\varrho^\bullet_{o,\iota(\bar{x})}(t_\bullet) = \varrho^\circ_{o,\bar{x}}(t_\circ)$). Since the maps $\overline{X}_{\sigma^\circ} \ni \bar{y} \mapsto \varrho^\circ_{o,\bar{y}}(t_\circ) \in X$ and $X \ni y \mapsto d_\bullet(y, \varrho^\circ_{o,\bar{x}}(t_\circ)) \in \mathbb{R}$ are continuous, the set $U := \{\bar{y} \in \overline{X}_{\sigma^\circ} : d_\bullet(\varrho^\circ_{o,\bar{y}}(t_\circ), \varrho^\circ_{o,\bar{x}}(t_\circ)) < \epsilon_\bullet\}$ is an open neighbourhood of $\bar{x}$ in $\overline{X}_{\sigma^\circ}$. Then, for any $\bar{y} \in U$, by Proposition 2.2 applied for the metric $d_\bullet$ and bicombing $\sigma^\bullet$ to the points $\varrho^\bullet_{o,\iota(\bar{x})}(t_\bullet)(= \varrho^\circ_{o,\bar{x}}(t_\circ))$ and $\varrho^\circ_{o,\bar{y}}(t_\circ)$, and radius $t_\bullet$, we obtain that $d_\bullet(\varrho^\bullet_{o,\iota(\bar{x})}(t_\bullet), \varrho^\bullet_{o,\iota(\bar{y})}(t_\bullet)) \leq 2d_\bullet(\varrho^\circ_{o,\bar{x}}(t_\circ), \varrho^\circ_{o,\bar{y}}(t_\circ)) < 2\epsilon_\bullet$. Similarly, the inverse ι^{-1} is continuous as well. $\square$

Now we are ready to prove the main theorem of this section.

Proof. (of Theorem 4.1 (Thm. IV)) Consider the complexes C^{90} and C^{45}, each

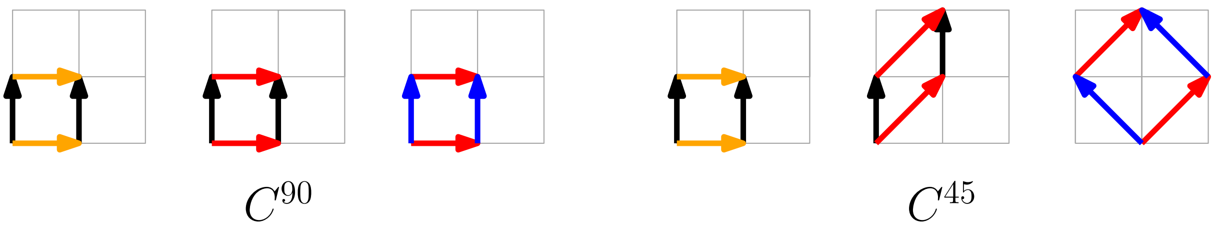

Figure 2: The tori glued to form complexes C^{90} and C^{45}, placed on the plane. The grey lines represent the $\mathbb{Z}\times\mathbb{Z}$–grid in $\mathbb{R}\times\mathbb{R}$.

consisting of appropriately glued three tori, as described on Figure 2. Let $X^{\alpha,p}$ for $\alpha \in \{45, 90\}$ and $p \in \{2, \infty\}$ be the universal cover of C^α with the metric being the extension of the local metric from C^α in the case when the underlying planes are endowed with the ℓ^p-metric. The fundamental group $G := \pi_1(C^{90})(= \pi_1(C^{45}))$ acts geometrically on each of these four spaces. Since each point in each of the spaces $X^{\alpha,\infty}$ (resp. $X^{\alpha,2}$) for

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$\alpha \in \{90, 45\}$ has a neighbourhood homeomorphic to some open ball in some of the 5 spaces considered in Lemma 4.7, the spaces $X^{\alpha,\infty}$ (resp. $X^{\alpha,2}$) are locally injective (resp. locally CAT(0)), thus by [Mie17, Theorem 1.2] (resp. by the Cartan–Hadamard Theorem for CAT(0) spaces, [BH99, Theorem II.4.1(2)]) they are injective (resp. CAT(0)); let $\sigma^{\alpha,\infty}$ (resp. $\sigma^{\alpha,2}$) be their convex bicombings. The classical result [CK00] gives that $\partial_{\sigma^{90,2}} X^{90,2}$ and $\partial_{\sigma^{45,2}} X^{45,2}$ are not homeomorphic. In order to finish the proof, we show that for each $\alpha \in \{90, 45\}$ we have that $\partial_{\sigma^{\alpha,\infty}} X^{\alpha,\infty} \cong \partial_{\sigma^{\alpha,2}} X^{\alpha,2}$. In view of Proposition 4.8, it suffices to show that $\operatorname{im} \sigma^{\alpha,\infty}_{ab} = \operatorname{im} \sigma^{\alpha,2}_{ab}$ for all $a, b \in X^\alpha := X^{\alpha,\infty} (= X^{\alpha,2})$. First, we check it locally. Observe that for each $x \in X^\alpha$ the space X^α may be identified locally around point x with some (subset of a) space M considered in Lemma 4.7 in such a way that for any $p \in \{2, \infty\}$ there exists $r_p > 0$ such that this identification gives an isometric identification of the ball $\overline{B}(x, r_p)$ in $X^{\alpha,p}$ with the ball $\overline{B}(x, r_p)$ in M^p. By conicality, for any $p \in \{2, \infty\}$ the ball $\overline{B}(x, r_p)$ is convex with respect to the $\mathfrak{ccc}$ bicombings on $X^{\alpha,p}$ and M^p, therefore the $\mathfrak{ccc}$ bicombings from these spaces restrict to $\mathfrak{ccc}$ bicombings on $\overline{B}(x, r_p)$; they must restrict to the same bicombing, as the balls in injective and CAT(0) spaces are injective and CAT(0), respectively, which implies that the $\mathfrak{ccc}$ bicombing on the ball $\overline{B}(x, r_p)$ in $X^{\alpha,p}$ is unique. Therefore, Lemma 4.7 implies that for any $a, b \in B_{X^{\alpha,\infty}}(x, r_\infty) \cap B_{X^{\alpha,2}}(x, r_2)$ we have that $\operatorname{im} \sigma^{\alpha,2}_{ab} = \operatorname{im} \sigma^{\alpha,\infty}_{ab}$. In the general case, by the above, the image of a (global) $\sigma^{\alpha,\infty}$-geodesic $\sigma^{\alpha,\infty}_{ab}$ is the image of some local CAT(0)-geodesic, which is also the unique global CAT(0)-geodesic $\sigma^{\alpha,2}_{ab}$ by [BH99, Proposition II.1.1.4(1, 2)] (or Proposition 4.6). $\square$

Remark 4.9. (i) One may construct complexes as on Figure 2 parametrised by the ℓ^2-angle $\alpha \in (0, \pi/2]$ between the **black** and red segment (see [CK00, Section 1.3]). Wilson [Wil05] improved the result of Croke and Kleiner by showing that for each pair of different angles $\alpha, \beta \in (0, \pi/2]$ the resulting boundaries (of the universal cover with the locally ℓ^2 path metric) are not homeomorphic, thus the fundamental group of (all) these complexes admits $2^{\aleph_0}$ pairwise non-homeomorphic CAT(0)-boundaries. However, the only pair of lines in a pair of ℓ^∞-planes, along which they can be glued to obtain an injective metric space, are two diagonal ones, or two horizontal-or-vertical ones — one may use the argument in [Mie15, Section 5] almost verbatim. Since injective metric spaces are locally injective, this prevents us from using Wilson's approach directly.

(ii) Since the construction of the spaces $X^{45,\infty}$ and $X^{90,\infty}$ consist in gluing injective planes along horizontal, vertical and diagonal lines, one may obtain graphs Γ^α for $\alpha \in \{45, 90\}$ by replacing each injective plane in the construction of $X^{\alpha,\infty}$ with a graph on $\mathbb{Z}^2$ with vertices $(x_1, y_1), (x_2, y_2) \in \mathbb{Z}^2$ connected by an edge iff $|x_1 - y_1|, |x_2 - y_2| \leq 1$ (by the graph Γ^α we mean the metric space with its underlying set consisting of just the vertices, with the edges used only to induce the path metric making the ends of each edge at distance 1 from each other) to prove an analogue of Theorem 4.1 for Helly groups. However, the graph Γ^{45} is not Helly. To see this, consider a pair of planes $P_i = \{(x,y)^{P_i} : x, y \in \mathbb{R}\}$ for $i = 1,2$ that are glued along their diagonals $\{(d,d)^{P_i} : d \in \mathbb{R}\}$ in the construction of $X^{45,\infty}$ (so that $(d,d)^{P_1} = (d,d)^{P_2}$ for all $d \in \mathbb{R}$). Consider the unit balls in Γ^{45} centred at points $a_1 = (-1,0)^{P_1}$, $a_2 = (1,2)^{P_1}$ and $a_3 = (1,0)^{P_2}$. They clearly intersect pairwise, however their intersection is empty. To see this, observe that the diagonal $D = \{(d,d)^{P_1} : d \in \mathbb{R}\}$ disconnects Γ^{45} so that a_1 and a_3 are in different connected components of $\Gamma^{45} \setminus D$. Therefore $\overline{B}_{\Gamma^{45}}(a_1,1) \cap \overline{B}_{\Gamma^{45}}(a_3,1) \subseteq D$, consequently $\overline{B}_{\Gamma^{45}}(a_1,1) \cap \overline{B}_{\Gamma^{45}}(a_3,1) = \{(0,0)^{P_1}\}$, but $(0,0)^{P_1} \notin \overline{B}_{\Gamma^{45}}(a_2,1)$.

5. Products and CAT(0) cube complexes with injective metrics

Let X be a cube complex. We denote by d_p^X for $p \in [1,\infty]$ the gluing metric on X arising from endowing each cube C of X with the ℓ^p-metric from the unit cube $[0,1]^{\dim C}$.

Remark 5.1. That d_p^X is indeed a metric, not just a pseudometric, is a consequence (cf. [BH99, Corollary I.7.10]) of the following observation: for each point x of a cube complex X there exists $\epsilon_x > 0$ such that for each cube C of X that contains x the d_p^C-distance (note that we do not restrict here d_p^X from X to C) from x to the faces of C that do not contain x is at least ϵ_x. Indeed, let $C(x)$ be the cube of X that contains x in its interior. If x is not a vertex of X, let $\epsilon_x > 0$ be the distance in $(C(x), d_p^{C(x)})$ from x to the faces of $C(x)$, otherwise put $\epsilon_x := 1$. Since every cube C containing x is a product of $C(x)$ with another cube, the observation follows.

In this section we build on the ideas introduced in Section 4 to prove the following theorem.

Theorem 5.2. *Let X be a locally finite CAT(0) cube complex of dimension at most 2. Let σ^p be the convex bicombing on (X, d_p) for $p \in \{2,\infty\}$. Then σ^2 and σ^∞ have the same trajectories.*

Remark 5.3. Below we justify correctness ((i) and (iii)) and further discuss ((ii) and (iii)) the above statement using results from literature.

Let X be a cube complex.

(i) If (X, d_2^X) is locally finite and CAT(0), then (X, d_∞^X) is injective. In the setting when X is a finite complex, one may refer to [Mie14, Theorems 1.4 and 1.3], or follow a

CAT(0) cube complexes–median cube complexes–collapsible polyhedra–injective metric spaces route, see [Che00; vdV98; MT83]. The case when X is locally finite may be then justified as follows: for each point x of the complex X consider the smallest subcomplex X_x of X that contains all of the cubes of X that contain x; the complex $(X_x, d_2^{X_x})$ is CAT(0) (this fact is discussed in detail in Remark 5.7) and finite (as X is locally finite); therefore $(X_x, d_\infty^{X_x})$ is injective; since the inclusion $(X_x, d_\infty^{X_x}) \hookrightarrow (X, d_\infty^X)$ is an isometry on a neighbourhood of x (recall the observation from Remark 5.1), the space (X, d_∞^X) is locally injective; the space X is also contractible, therefore (X, d_∞^X) is injective by [Mie17, Theorem 1.2].

(ii) Injectivity of (X, d_∞) implies CAT(0)-ness of (X, d_2) — see [Mie14, Theorem 1.2], which uses the Link Condition, [Gro87, the 4.2.C that follows 4.2.D].

(iii) The discussion regarding uniqueness of bicombings described in Remark 4.2 applies to (X, d_2) if it is CAT(0), and to (X, d_∞) if it is injective and locally finite. As in Section 4, further in this section we usually use these uniqueness results without mentioning.

In view of Proposition 4.8, we have the following corollary.

Corollary 5.4 (Theorem V). *Let X be a locally finite CAT(0) cube complex of dimension at most 2. Let σ^p be the convex bicombing on (X, d_p) for $p \in \{2, \infty\}$. Then the identity of X extends to a homeomorphism between $\overline{X}_{\sigma^2}$ and $\overline{X}_{\sigma^\infty}$, in particular the boundaries $\partial_{\sigma^2} X$ and $\partial_{\sigma^\infty} X$ are homeomorphic.*

We begin the preparations for the proof of Theorem 5.2 with some general definitions and observations.

Let (X, d_X) and (Y, d_Y) be metric spaces. By the *ℓ^p-product* $X \times_p Y$ we mean the Cartesian product $X \times Y$ endowed with the metric

$$d_{X\times_p Y}((x_1, y_1), (x_2, y_2)) = \big\|\big(d_X(x_1, x_2), d_Y(y_1, y_2)\big)\big\|_p.$$

If the spaces X, Y admit bicombings σ^X, σ^Y, respectively, then the *product bicombing* $\sigma^X \otimes \sigma^Y$ is defined to be the map

$$(X \times Y) \times (X \times Y) \times [0, 1] \ni \big((x_1, y_1), (x_2, y_2), t\big) \longmapsto \big(\sigma^X_{x_1,x_2}(t), \sigma^Y_{y_1,y_2}(t)\big) \in X \times Y.$$

Remark 5.5. Let (X, d_X) and (Y, d_Y) be metric spaces that admit bicombings σ^X and σ^Y, respectively. Then, the product bicombing $\sigma^X \otimes \sigma^Y$ is consistent, conical, convex, and reversible iff both of the bicombings σ^X and σ^Y are consistent, conical, convex, and reversible, respectively. The $\Longrightarrow$ implication follows as the spaces X and Y embed into $X \times_p Y$ as sections, to which the product bicombing $\sigma^X \otimes \sigma^Y$ restricts as σ^X and

σ^Y, respectively. Regarding the $\Longleftarrow$ implication, the claim regarding consistency and reversibility follows directly; the claim regarding conicality and convexity follows from a direct calculation involving the Minkowski's inequality.

Lemma 5.6. *Let $\{(X_i, d_i) : i \in I\}$ be a family of metric spaces that are glued together to form a (metric) space (X, d_X), where d_X is the gluing metric, and denote the gluing map $\pi\colon \bigsqcup_{i\in I} X_i \to X$. Let Y be a geodesic metric space and $p \in [1, \infty]$. Then gluing and taking the ℓ^p-product commute, namely: the gluing metric d_{gl} on $X \times Y$ arising from the gluing map $\pi \times \mathrm{id}_Y\colon \bigsqcup_{i\in I}(X_i \times Y, d_{X_i\times_p Y}) \to X \times Y$ is the same as the ℓ^p-metric $d_{X\times_p Y}$ on $X \times Y$.*

Proof. Let $x, x' \in X$ and $y, y' \in Y$. Consider an $((x, y),(x', y'))$–gluing path $P = ((x, y) = (x_0, y_0), \dots, (x_n, y_n) = (x', y'))$ in $X \times Y$ and $i_j \in I$ for $j = 1, \dots, n$, such that $x_{j-1}, x_j \in X_{i_j}$. Then P induces an (x, x')–gluing path $P_X = (x_0, \dots, x_n)$ in X. The length $\mathrm{len}(P)$ satisfies

$$
\begin{aligned}
\mathrm{len}(P) = \sum_{j=1}^{n} \big\| \big(d_{X_{i_j}}(x_{j-1}, x_j), d_Y(y_{j-1}, y_j)\big) \big\|_p \\
&\geq \Big\| \Big(\sum_{j=1}^{n} d_{X_{i_j}}(x_{j-1}, x_j), \sum_{j=1}^{n} d_Y(y_{j-1}, y_j)\Big) \Big\|_p \\
&= \Big\| \Big(\mathrm{len}(P_X), \sum_{j=1}^{n} d_Y(y_{j-1}, y_j)\Big) \Big\|_p \geq \big\| \big(d_X(x_0, x_n), d_Y(y_0, y_n)\big) \big\|_p,
\end{aligned}
$$

therefore $d_{\mathrm{gl}}((x, y),(x', y')) \geq d_{X\times_p Y}((x, y),(x', y'))$.

For the other inequality, let $P_X = (x = x_0, \dots, x_n = x')$ be an (x, x')–gluing path of non-zero length in X such that for every $1 \leq j \leq n$ the elements x_{j-1}, x_j belong to X_{i_j} for some $i_j \in I$. Given any $y, y' \in Y$, choose a sequence of points $P_Y = (y = y_0, y_1, \dots, y_n = y')$ in Y such that $d_Y(y_{j-1}, y_j) = d_{X_{i_j}}(x_{j-1}, x_j) \cdot d_Y(y, y')/\mathrm{len}(P_X)$ for $1 \leq j \leq n$. Then $P = ((x_0, y_0), \dots, (x_n, y_n))$ is an $((x, y),(x', y'))$–gluing path in $X \times Y$ that satisfies $\mathrm{len}(P) = \|(\mathrm{len}(P_X), d_Y(y, y'))\|_p$, and the other inequality follows. $\square$

Below we present and discuss several definitions concerning local fragments of a cube complex.

Let X be a cube complex. For any cube C of X we denote by X_C the smallest subcomplex of X containing all of the cubes of X that contain C. In further text, we usually consider X_C with the metrics $d_2^{X_C}$ and $d_\infty^{X_C}$ (discarding the cubes of X not belonging to X_C — in particular we do not restrict the metric from X to X_C). We define $C^\perp$ to be any of the (identical) subcomplexes of X_C such that X_C is the product $C \times C^\perp$.

Remark 5.7. If C is a cube of a CAT(0) cube complex (X, d_2^X), then the above-defined cube complex $(X_C, d_2^{X_C})$ is also CAT(0): since X_C is contractible, it is sufficient to check that it satisfies the Link Condition. Let c be any vertex of C. For each vertex v of X_C there exists the smallest cube C_v containing C and v; denote by Δ_v the set of neighbours of c (in the 1-skeleton of X) that belong to C_v. Consider a vertex a of X_C and neighbours $v_1, \ldots, v_n$ of a in X_C such that for all $1 \le i < j \le n$ the vertices v_i, v_j span an edge in the link of a in X_C, i.e. there exists a cube C_{ij} in X containing a, v_i, v_j and C. Then the cube C_{ij} must contain C_a, C_{v_i} and C_{v_j}, hence each pair of vertices from $\Delta_a \cup \Delta_{v_i} \cup \Delta_{v_j}$ is connected by an edge in the link of c. Therefore, every pair of vertices of the set $\Delta_a \cup \bigcup\{\Delta_{v_i} : 1 \le i \le n\}$ spans an edge in the link of c, and by the Link Condition for c in X, there exists a cube in X that contains a, C and all of the v_i, as required.

Given a point $x \in X$, let $C(x)$ be the cube in X that contains x in its interior (we use the convention that the interior of a 0-cube is the 0-cube itself). We then use the shorter notation X_x for $X_{C(x)}$ and say that the point x is of *type* $(\dim X_x - \dim C(x), \dim C(x))$. (Note that the first coordinate of type is equal to $\dim C(x)^\perp$. The type with the lexicographic order may be considered to be a measure of complexity of a local neighbourhood of x in the cube complex X.)

We say that a locally finite CAT(0) cube complex X has the *property* Traj if the convex d_2^X-bicombing σ^2 on X and the convex d_∞^X-bicombing σ^∞ on X have the same trajectories, i.e. $\operatorname{im}\sigma^2_{xx'} = \operatorname{im}\sigma^\infty_{xx'}$ for all $x, x' \in X$; and say that X has the *property* Par if furthermore the parametrisations agree, i.e. the bicombings σ^2 and σ^∞ are equal.

The main technical lemma used in the proof of Theorem 5.2 to carry an induction over the type is as follows.

Lemma 5.8. *Let X be a locally finite CAT(0) cube complex and $x \in X$.*

(i) If x is of type $(0,0)$, $(0,1)$ or $(1,0)$, i.e. $\dim X_x \le 1$, then X_x satisfies Par*.*

(ii) If $C(x)^\perp$ satisfies Par*, then X_x also satisfies* Par*.*

(iii) If x is of type $(n,0)$, i.e. x is a vertex and $\dim X_x = n$, for some $n > 0$, and for all $x_\circ \ne x$ belonging to the interior of a cube containing x the cube complex $X_{x_\circ}$ satisfies Traj*, then X_x also satisfies* Traj*.*

Proof. (i) Type $(0,0)$. In this case X_x is a single point. The claim follows.

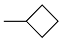
Type $(0,1)$. In this case X_x is a unit interval I, on which the ℓ^2-metric d_2^I coincides with the ℓ^∞-metric d_∞^I, therefore the convex d_2^I-bicombing on I and the convex d_∞^I-bicombing on I are equal.

Type $(1,0)$. In this case the complex X_x consists of several copies of the unit interval I glued in the common vertex x. Since $d_2^I = d_\infty^I$, also the equality $d_2^{X_x} = d_\infty^{X_x}$ holds, therefore the convex $d_2^{X_x}$-bicombing on X_x and the convex $d_\infty^{X_x}$-bicombing on X_x are equal.

(ii) By Lemma 5.6 the complex $(X_x, d_p^{X_x})$ is the ℓ^p-product of $(C(x)^\perp, d_p^{C(x)^\perp})$ with the $\dim C(x)$–fold ℓ^p-product of the unit interval (I, d_p^I). By the assumption and (i), the complex $C(x)^\perp$ and the interval I, respectively, satisfy Par. For $p \in \{2,\infty\}$, denote by $\sigma^{p,\perp}$ the convex $d_p^{C(x)^\perp}$-bicombing on $C(x)^\perp$, and by σ^I the bicombing on the unit interval. We then have the equality $\sigma^{2,\perp} \otimes (\sigma^I)^{\otimes \dim C(x)} = \sigma^{\infty,\perp} \otimes (\sigma^I)^{\otimes \dim C(x)}$. The bicombing $\sigma^{p,\perp} \otimes (\sigma^I)^{\otimes \dim C(x)}$ is the convex $d_p^{X_x}$-bicombing on X_x, see Remark 5.5, which finishes the proof.

(iii) For $p \in \{2,\infty\}$, denote by σ^p the convex $d_p^{X_x}$-bicombing on X_x. Observe that for any point $x_\circ \neq x$ belonging to the interior of a cube incident to x, the complex $X_{x_\circ}$ is the subcomplex $(X_x)_{x_\circ}$ of X_x. Consider $x' \in X_x \setminus \{x\}$ and let $x'_\circ$ be any point from the interior of the smallest cube containing both x and x'. Since some open neighbourhood of x' in X_x is contained in $X_{x'_\circ}$, and the inclusion $(X_{x'_\circ}, d_p^{X_{x'_\circ}}) \hookrightarrow (X_x, d_p^{X_x})$ is an isometry on a neighbourhood of x' (recall Remark 5.1), and for all metrics d the d-balls are convex with respect to conical d-bicombings (here we consider $d = d_p^{X_x}$ and $d = d_p^{X_{x'_\circ}}$), by uniqueness of bicombings (see Remark 5.3(iii)) there exists an open neighbourhood $U_{x'}$ of x' in X_x such that $\operatorname{im} \sigma^2_{ab} = \operatorname{im} \sigma^\infty_{ab}$ for all $a, b \in U_{x'}$.

Now consider any $a, b \in X_x$. We shall prove that $\sigma^2_{ab} = \sigma^\infty_{ab}$. The case of $a = x = b$ is clear. Next, we consider the case when $a \neq x = b$. Let $a_\circ$ belong to the interior of the smallest cube that contains both a and x. Let γ be the (straight-line) geodesic from the convex $d_2^{C(a_\circ)}$-bicombing on $C(a_\circ)$ that begins in a and ends in x. Since the inclusion $(C(a_\circ), d_2^{C(a_\circ)}) \hookrightarrow (X_x, d_2^{X_x})$ is a local isometry on the interior of the cube $C(a_\circ)$ (recall Remark 5.1), the restrictions $\gamma|_{[s,t]}$ are local σ^2-geodesics for $0 < s \leq t < 1$ (recall that CAT(0)-geodesics are unique). Therefore, by considering the neighbourhoods $U_{x'}$, one obtains that for all $0 < s \leq t < 1$ the image $\operatorname{im} \gamma|_{[s,t]}$ is the image of a local σ^∞-geodesic; then, by Proposition 4.6, we have that these local σ^p-geodesics are (global) σ^p-geodesics, hence $\operatorname{im} \sigma^2_{\gamma(s),\gamma(t)} = \operatorname{im} \gamma|_{[s,t]} = \operatorname{im} \sigma^\infty_{\gamma(s),\gamma(t)}$; finally, passing to the limits $s \to 0$ and $t \to 1$, we obtain that $\operatorname{im} \sigma^2_{ax} = \operatorname{im} \gamma = \operatorname{im} \sigma^\infty_{ax}$, as required. Likewise follows the case when $a = x \neq b$. In the case when $a \neq x \neq b$, we consider two subcases. If for some $p \in \{2,\infty\}$ there exists a local σ^p-geodesic γ from a to b in X_x that omits x, then, by considering the neighbourhoods $U_{x'}$, one obtains that the image $\operatorname{im} \gamma$ is both the image of a local σ^2-geodesic and the image of a local σ^∞-geodesic, therefore by Proposition 4.6 $\operatorname{im} \sigma^2_{ab} = \operatorname{im} \gamma = \operatorname{im} \sigma^\infty_{ab}$. Otherwise, for both choices of $p \in \{2,\infty\}$ the σ^p-geodesic σ^p_{ab} passes through x and must be the concatenation of the geodesics σ^p_{ax} and σ^p_{xb}; the image of each of these geodesics does not depend on p, as it was discussed in the case when $a \neq x = b$. $\square$

Remark 5.9. Denote by F the space consisting of a 1-cube I glued with a 2-cube $\square$ in a vertex: . Observe that, even though I and $\square$ satisfy Par, the lengths of the same bicombing-geodesic in $(\square, d_2)$ and $(\square, d_\infty)$ almost always differ, which leads to the following.

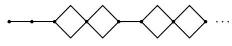
(i) By considering the space F, one obtains that the ($\mathsf{Par} \Rightarrow \mathsf{Par}$)-version of Lemma 5.8(iii) does not hold.

(ii) By considering the product of F with the unit interval, one obtains that the ($\mathsf{Traj} \Rightarrow \mathsf{Traj}$)-version of Lemma 5.8(ii) does not hold.

Now we are ready to prove the main theorem of this section.

Proof. (of Theorem 5.2) Denote by $\mathsf{Traj}(n,m)$ the property that for every CAT(0) cube complex Y and every point $y \in Y$ of type (n,m) the complex Y_y satisfies Traj; likewise for Par. Lemma 5.8(i) states that $\mathsf{Par}(0,0)$, $\mathsf{Par}(0,1)$ and $\mathsf{Par}(1,0)$ hold. Next, observe that for any element y of a cube complex Y, upon denoting by $y^\perp$ the only element of the intersection $C(y) \cap C(y)^\perp$ (which is a single vertex), the equality $C(y)^\perp = (C(y)^\perp)_{y^\perp}$ holds (any cube $C^\downarrow$ in $C(y)^\perp$ is the intersection $C \cap C(y)^\perp$ for some cube C of Y containing y; then the cube C also contains $C(y)$, to which $y^\perp$ belongs; hence $y^\perp$ belongs to $C \cap C(y)^\perp = C^\downarrow$); and, upon denoting by (n,m) the type of y in Y, the type of $y^\perp$ in the complex $C(y)^\perp$ is $(\dim(C(y)^\perp)_y - 0, 0) = (\dim C(y)^\perp, 0) = (\dim Y_y - \dim C(y), 0) = (n, 0)$. Therefore, by Lemma 5.8(ii), $\mathsf{Par}(0,2)$ and $\mathsf{Par}(1,1)$ follow from $\mathsf{Par}(0,0)$ and $\mathsf{Par}(1,0)$, respectively. Finally, it follows from Lemma 5.8(iii) that $\mathsf{Traj}(2,0)$ holds, as for a 2-dimensional cube complex Y and $y \in Y$, for all $y_\circ \neq y$ belonging to the interior of a cube $C(y_\circ)$ in Y containing y one has that the type (n,m) of $y_\circ \in Y$ satisfies $n+m = \dim Y_{y_\circ} \leq \dim Y = 2$ and $m = \dim C(y_\circ) > 0$.

Then, as the type (n,m) of each element of X satisfies $n + m \leq \dim X \leq 2$, and for $p \in \{2, \infty\}$ the d_p-balls are σ^p-convex, and for $x \in X$ the inclusion $(X_x, d_p^{X_x}) \hookrightarrow (X, d_p^X)$ is a local isometry on a neighbourhood of x (recall Remark 5.1), we have a family of open subsets $\{U_x \subseteq X : x \in X\}$ such that $\operatorname{im} \sigma^2_{ab} = \operatorname{im} \sigma^\infty_{ab}$ for all $x \in X$ and $a, b \in U_x$. Therefore by Proposition 4.6 the bicombings σ^p have the same trajectories. $\square$

Remark 5.10. (i) The product of the space F from Remark 5.9 with the unit interval is an example of a finite CAT(0) cube complex of dimension 3 for which the conclusion of Theorem 5.2 does not hold. Furthermore, its underlying idea can be used in the following construction of a locally finite CAT(0) cube complex P of dimension 3 such that the identity map $(P, d_2) \to (P, d_\infty)$ does not extend continuously to a map between compactifications (which is the first conclusion of the Corollary 5.4). Let $x_1, x_2, x_3, \ldots$ be such that $x_1 = 1$, $x_n \in \{1, \sqrt{2}\}$, and the sequence $(\sum_{i=1}^n x_i)/n$ is not convergent. Consider cubes $C_1, C_2, C_3, \ldots$ such that C_n is a 1-cube iff $x_n = 1$, and C_n is a 2-cube iff $x_n = \sqrt{2}$. Let N be the cube complex similar to , obtained from the sequence of cubes (C_n) — more precisely, we denote by N the result of gluing the cubes C_n in such a way that the cubes C_i and C_j are glued to each other iff $|i - j| = 1$, every gluing is a gluing along a single vertex, and if C_n is a 2-cube, then the two vertices participating in the gluing are on the opposite sides of a diagonal D_n of C_n. Denote by L the half-line in N obtained by taking the union of all of the cubes C_n that are 1-cubes with all of the diagonals D_n (of these of the C_n that are 2-cubes). Let P be the cube complex resulting from taking the product of N with the $\mathbb{R}$-line R subdivided into unit intervals.

By applying Lemma 4.3 to the symmetry along the half-plane $H := L \times R$, one obtains that that H is σ^2-convex and σ^∞-convex in P. For $p \in \{2, \infty\}$, the half-line $(L, d_p^N|_{L \times L})$ may be isometrically identified with the half-line $[0, \infty)$ (with the usual metric), giving rise to an isometric identification of the half-plane (H, d_p^H) with the ℓ^p-product $[0, \infty) \times_p \mathbb{R}$ (recall Lemma 5.6); below we use this identification by writing $(x, y) \in (H, d_p^H)$ for $x \geq 0$ and $y \in \mathbb{R}$. Upon these identifications, the identity of H sends the point $(\sum_{i=1}^n x_i, y) \in (H, d_2^H)$ to $(n, y) \in (H, d_\infty^H)$, where $n \in \mathbb{N}$ and $y \in \mathbb{R}$. Let $a \in (-1/\sqrt{2}, 1/\sqrt{2}) \setminus \{0\}$. The map $[0, \infty) \ni t \mapsto (t, at) \in (H, d_2^H)$ is a σ^2-ray in H (thus, in P), and the points $(\sum_{i=1}^n x_i, a \cdot \sum_{i=1}^n x_i) \in (H, d_2^H)$ for $n \in \mathbb{N}$ are mapped by the identity to the points $(n, a \cdot \sum_{i=1}^n x_i) \in (H, d_\infty^H)$; the σ^∞-geodesic from $(0, 0) \in (H, d_\infty^H)$ to the point $(n, a \cdot \sum_{i=1}^n x_i) \in (H, d_\infty^H)$ passes through the point $(1, a \cdot (\sum_{i=1}^n x_i)/n) \in (H, d_\infty^H)$. Since $|a| \in (0, \sqrt{2})$, we have that $|a \cdot (\sum_{i=1}^n x_i)/n| \leq |a|\sqrt{2} \leq 1$, so this point is at d_∞^H-distance 1 from $(0, 0)$. By the assumption on the sequence (x_n), the sequence $(1, a \cdot (\sum_{i=1}^n x_i)/n)$ is not convergent, therefore $(n, a \cdot \sum_{i=1}^n x_i) \in (H, d_\infty^H)$ is also not a convergent sequence in the compactification $\overline{P}_{\sigma^\infty}$, while it is convergent in the compactification $\overline{P}_{\sigma^2}$.

We do not know if the 'in particular' part of Corollary 5.4 may not hold for a locally finite CAT(0) cube complex. (In the above example, both of the boundaries $\partial_{\sigma^2} P$ and $\partial_{\sigma^\infty} P$ are homeomorphic to an interval, as may be seen using Lemma 5.6, the fact that each ray in N is contained in L, and Proposition 5.11 below.)

(ii) The local-to-global approach from the above proof of Theorem 5.2 works not only for (locally finite CAT(0)) cube complexes of dimension at most 2, however such extensions would lead to making the statement of Theorem 5.2 more technical. For instance, in the induction in the 'local' part of this proof, instead of considering only the very general $\mathsf{Traj}(n,m)$ and $\mathsf{Par}(n,m)$ properties (note that we have, in fact, proved $\mathsf{Par}(0,m)$ and $\mathsf{Par}(1,m)$ for arbitrary m, and $\mathsf{Traj}(2,0)$), one may start with a particular cube complex X, consider which subcomplexes of X are needed to carry the induction, and prove Traj or Par only for these — this approach works e.g. if the complex X has no elements of type (n,m) with $n \geq 2$ and $m \geq 1$.

In the course of this section we have made enough preparations to state the following proposition.

Proposition 5.11. *Let $p \in [1,\infty]$, and assume that proper metric spaces (X, d_X) and (Y, d_Y) admit* $\mathfrak{ccc}$ *bicombings σ^X and σ^Y, respectively. Then the boundary $\partial_{\sigma^X \otimes \sigma^Y} X \times_p Y$ is homeomorphic to the join $\partial_{\sigma^X} X * \partial_{\sigma^Y} Y$ of $\partial_{\sigma^X} X$ and $\partial_{\sigma^Y} Y$.*

Proof. First, assume that the space Y is compact; then $\partial_{\sigma^Y} Y = \emptyset$, and we need to show that the boundary $\partial_{\sigma^X \otimes \sigma^Y} X \times_p Y$ is homeomorphic to $\partial_{\sigma^X} X$. Pick a basepoint $o = (o_X, o_Y) \in X \times Y$. By the definition of the product bicombing, the set $X \times \{o_Y\}$ is $(\sigma^X \otimes \sigma^Y)$-convex in $X \times Y$, hence the map sending a σ^X-ray ξ^X based in o_X to the $(\sigma^X \otimes \sigma^Y)$-ray $t \mapsto (\xi^X(t), o_Y)$ induces a homeomorphic embedding of $\partial_{\sigma^X} X$ into $\partial_{\sigma^X \otimes \sigma^Y} X \times_p Y$. We show that it is onto by showing that every $(\sigma^X \otimes \sigma^Y)$-ray γ based in o has its image contained in $X \times \{o_Y\}$. Since Y is compact, there exists a constant $D > 0$ such that $Y \subseteq B_Y(o_Y, D)$. Then, for each $n \in \mathbb{N}$ there exist $a_n \in X \times \{o_Y\}$ such that $d_{X \times_p Y}(a_n, \gamma(n)) < D$. Proposition 2.2 then implies that for all $n \geq r > 0$ we have that

$$d_{X \times_p Y}(X \times \{o_Y\}, \gamma(r)) \leq d_{X \times_p Y}(\varrho_{o,a_n}(r), \gamma(r)) \leq 2D \cdot r/n,$$

which tends to 0 as $n \to \infty$; therefore $\gamma(r) \in X \times \{o_Y\}$, and $\operatorname{im} \gamma \subseteq X \times \{o_Y\}$. Likewise follows the case when X is compact.

Now we proceed to the main case. Assume that X are Y are non-compact. Then, as the spaces X and Y are proper, the compactifications $\overline{X}_{\sigma^X}$ and $\overline{Y}_{\sigma^Y}$ are compact, and the boundaries $\partial_{\sigma^X} X$ and $\partial_{\sigma^Y} Y$ are non-empty. Let $\daleth := \{(a,b) \in \mathbb{R}^2 : a, b \geq 0, \|(a,b)\|_p = 1\}$. Consider the map q given by

$$\partial_{\sigma^X} X \times \partial_{\sigma^Y} Y \times \daleth \ni \big([\xi_X], [\xi_Y], (a,b)\big) \mapsto \big[t \mapsto \big(\xi_X(at), \xi_Y(bt)\big)\big] \in \partial_{\sigma^X \otimes \sigma^Y} X \times_p Y.$$

One may check that q is well-defined, i.e. the map $t \mapsto (\xi_X(at), \xi_Y(bt))$ is a $(\sigma^X \otimes \sigma^Y)$-ray for all choices of a σ^X-ray ξ^X, a σ^Y-ray ξ^Y and $(a,b) \in \daleth$, whose asymptotic class does not change when ξ^X or ξ^Y is replaced with any other element of the asymptotic class of ξ^X or ξ^Y, respectively.

Let π be the quotient map of the equivalence relation $\sim_*$ on the set $\partial_{\sigma^X} X \times \partial_{\sigma^Y} Y \times \daleth$, where $\sim_*$ is — upon an identification of $\daleth$ with $[0,1]$ — the equivalence relation standardly used to define the join of the spaces $\partial_{\sigma^X} X$ and $\partial_{\sigma^Y} Y$, i.e. $\sim_*$ is the smallest equivalence relation such that $(x, y, (1,0)) \sim_* (x, y', (1,0))$ and $(x, y, (0,1)) \sim_* (x', y, (0,1))$ for all $x, x' \in \partial_{\sigma^X} X$ and $y, y' \in \partial_{\sigma^Y} Y$. Observe that for any pair ξ, ζ of σ^X-rays or of σ^Y-rays originating from a common basepoint, and numbers $a, a' > 0$, the function $[0,\infty) \ni t \mapsto d(\xi(at), \zeta(a't))$ is zero iff $a = a'$ and $\xi = \zeta$, or $a = a' = 0$. This observation implies that the map q (setwise) factors (in a unique way) as $q = q_\pi \pi$, with the map $q_\pi \colon \partial_{\sigma^X} X * \partial_{\sigma^Y} Y \to \partial_{\sigma^X \otimes \sigma^Y} X \times_p Y$ being one-to-one. We shall show that q_π is a homeomorphism: since the set $\partial_{\sigma^X} X \times \partial_{\sigma^Y} Y \times \daleth$ is compact, and we have already shown that q_π is one-to-one, it is sufficient to show that q_π is a continuous surjection.

The surjectivity of q_π is equivalent to the surjectivity of q. We check the latter below. Take a $(\sigma^X \otimes \sigma^Y)$-ray (γ^X, γ^Y) in $X \times_p Y$. By the definition of the product bicombing, we have that $\gamma^X(\alpha t) = \gamma^X|_{[0,t]}(\alpha t) = \sigma_{\gamma^X(0)\gamma^X(t)}(\alpha)$ for all $t \geq 0$ and $0 \leq \alpha \leq 1$; therefore for all $t > 0$ and $0 \leq s \leq s' \leq t$ we have that $\operatorname{im} \gamma^X|_{[s,s']} = \operatorname{im} \sigma_{\gamma^X(s),\gamma^X(s')}$ and $d_X(\gamma^X(s), \gamma^X(s')) = (s' - s)a_t$ (where $a_t = d_X(\gamma^X(0), \gamma^X(t))/t$). Dividing the second of these equalities by $(s' - s)$, we see that a_t is a constant independent of t — denote it by a; therefore γ^X satisfies $\operatorname{im} \gamma^X|_{[s,s']} = \operatorname{im} \sigma_{\gamma^X(s),\gamma^X(s')}$ and $d(\gamma^X(s), \gamma^X(s')) = (s' - s)a$ for all $s' \geq s \geq 0$. Hence: if $a \neq 0$, then $\xi^X \colon [0,\infty) \to X$ given by $\xi^X(t) = \gamma^X(a^{-1}t)$ is a σ^X-ray; and if $a = 0$, then γ^X is constant, so any σ^X-ray ξ^X originating in $\gamma^X(0)$ satisfies $\gamma^X(t) (= \gamma^X(0)) = \xi^X(at)$ for all $t \geq 0$. Similarly, one obtains that there exists a σ^Y-ray ξ^Y such that $\gamma^Y(t) = \xi^Y(bt)$ for all $t \geq 0$, where $b = d_Y(\gamma^Y(0), \gamma^Y(1))$. The speed parameters a, b satisfy

$$\begin{aligned} 1 = d_{X \times_p Y}\big((\gamma^X(0), \gamma^Y(0)), (\gamma^X(1), \gamma^Y(1))\big) \\ = \big\|\big(d_X(\gamma^X(0), \gamma^X(1)), d_Y(\gamma^Y(0), \gamma^Y(1))\big)\big\|_p = \|(a,b)\|_p. \end{aligned}$$

The surjectivity of q follows.

The continuity of q_π is equivalent to the continuity of q, which we check below at each point $(\bar{x}, \bar{y}, (a,b)) \in \partial_{\sigma^X} X \times \partial_{\sigma^Y} Y \times \daleth$. Fix a basepoint $o = (o_X, o_Y) \in X \times Y$. Consider any $R, \epsilon > 0$. Then for any $\delta > 0$, $\bar{x}' \in U_{o_X}(\bar{x}, R, \delta) \cap \partial_{\sigma^X} X$, $\bar{y}' \in U_{o_Y}(\bar{y}, R, \delta) \cap \partial_{\sigma^Y} Y$, and $(a', b') \in \daleth$ with $|a' - a|, |b' - b| < \delta$, we have

$$\begin{aligned} d_X(\varrho^X_{o_X,\bar{x}}(aR), \varrho^X_{o_X,\bar{x}'}(a'R)) \leq d_X(\varrho^X_{o_X,\bar{x}}(aR), \varrho^X_{o_X,\bar{x}}(a'R)) + d_X(\varrho^X_{o_X,\bar{x}}(a'R), \varrho^X_{o_X,\bar{x}'}(a'R)) \\ \leq \delta R + a'R\delta \leq \delta R + \delta R = 2\delta R. \end{aligned}$$

Since a similar reasoning applies in Y, we jointly have

$$d_{X \times_p Y}\big(\varrho^{X\times Y}_{o,q(\bar{x},\bar{y},(a,b))}(R), \varrho^{X\times Y}_{o,q(\bar{x}',\bar{y}',(a',b'))}(R)\big) \leq 2^{1/p} \cdot 2\delta R,$$

which is smaller than ϵ for sufficiently small δ. $\square$

6. Quasisymmetric structure on boundary

Let (X,d) be a complete metric space that admits a $\mathfrak{ccc}$ bicombing σ. In this section we define a metric $d_{o,C}$ on the boundary $\partial_\sigma X$ such that if a group G acts on X via isometries and σ is G-equivariant, then the induced action on $(\partial_\sigma X, d_{o,C})$ is via quasisymmetries (see definitions below). Similar properties of a similar metric have been studied in the case of CAT(0) spaces in [Mor16, Section 3.1] and [OŚ15, Proposition 9.6(1)], where, basically, only the $\mathfrak{ccc}$-ness of the CAT(0)-bicombing is used, and we reformulate and adapt the proofs to our setting.

For metric spaces $(X_1,d_1),(X_2,d_2)$, a map $f\colon (X_1,d_1)\to(X_2,d_2)$ is a *quasisymmetry* if f is not constant and there exists a homeomorphism $\eta\colon[0,\infty)\to[0,\infty)$ such that for all $x,y,z\in X_1$ and $t\geq 0$, if $d_1(x,y)\leq td_1(x,z)$, then $d_2(f(x),f(y))\leq \eta(t)d_2(f(x),f(z))$. By [Hei01, Proposition 10.6], quasisymmetries are closed under composition, each quasisymmetry is one-to-one, and the inverse (defined on its image) of a quasisymmetry is also a quasisymmetry.

Let $C>0$ and $o\in X$. For $\bar x_1,\bar x_2\in\partial_\sigma X$ such that $\bar x_1\neq\bar x_2$, define $d_{o,C}(\bar x_1,\bar x_2)=t^{-1}$, where t is the unique number such that $d(\varrho_{o,\bar x_1}(t),\varrho_{o,\bar x_2}(t))=C$ (recall Proposition 2.1(i)), and let $d_{o,C}(\bar x,\bar x)=0$ for all $\bar x\in\partial_\sigma X$.

Proposition 6.1. *Let (X,d) be a complete metric space that admits a $\mathfrak{ccc}$ bicombing σ. Then the following hold for every $C,C'>0$ and $o,o'\in X$.*

(i) The map $d_{o,C}$ is a metric.

(ii) The topology induced by $d_{o,C}$ coincides with the topology on $\partial_\sigma X$ defined in Section 2.

(iii) The identity map $\mathrm{id}\colon(\partial_\sigma X,d_{o,C})\to(\partial_\sigma X,d_{o,C'})$ *is a quasisymmetry.*

(iv) The identity map $\mathrm{id}\colon(\partial_\sigma X,d_{o,C})\to(\partial_\sigma X,d_{o',C})$ *is a quasisymmetry.*

(v) Let G act on X via isometries in such a way that σ is G-equivariant. Then the extension of the action of each element of G to $\overline{X}_\sigma$ restricts to a quasisymmetry of $(\partial_\sigma X,d_{o,C})$.

Proof. (i) Let $\bar x_1,\bar x_2,\bar x_3\in\partial X$. The function $d_{o,C}$ clearly satisfies $d_{o,C}(\bar x_1,\bar x_2)=0$ iff $\bar x_1=\bar x_2$, and is symmetric, so it remains to show that $d_{o,C}$ satisfies the triangle inequality. Suppose without loss of generality that $d_{o,C}(\bar x_1,\bar x_3)\geq d_{o,C}(\bar x_1,\bar x_2),d_{o,C}(\bar x_2,\bar x_3)$, and let $t_{ij}:=d_{o,C}(\bar x_i,\bar x_j)^{-1}$. We have that

$$\begin{aligned}C&=d(\varrho_{o,\bar x_1}(t_{13}),\varrho_{o,\bar x_3}(t_{13}))\leq d(\varrho_{o,\bar x_1}(t_{13}),\varrho_{o,\bar x_2}(t_{13}))+d(\varrho_{o,\bar x_2}(t_{13}),\varrho_{o,\bar x_3}(t_{13}))\\&\leq(t_{13}/t_{12})d(\varrho_{o,\bar x_1}(t_{12}),\varrho_{o,\bar x_2}(t_{12}))+(t_{13}/t_{23})d(\varrho_{o,\bar x_2}(t_{23}),\varrho_{o,\bar x_3}(t_{23}))=Ct_{13}(t_{12}^{-1}+t_{23}^{-1}),\end{aligned}$$

where the last inequality follows from conicality of σ. Therefore $t_{12}^{-1} \leq t_{13}^{-1} + t_{23}^{-1}$, and the claim follows.

(ii) Observe that by the definition, for any $\epsilon, r > 0$ and $\bar{x}_1 \in \partial X$ we have that $B_{d_{o,C}}(\bar{x}_1, \epsilon) = U_o(\bar{x}_1, 1/\epsilon, C) \cap \partial_\sigma X$. To finish the proof of the claim, consider $\bar{x}_1, \bar{x}_2 \in \partial_\sigma X$ and $R, \epsilon > 0$ such that $\bar{x}_2 \in U_o(\bar{x}_1, R, \epsilon) \cap \partial_\sigma X$. There exists ϵ' such that $C > \epsilon' > 0$ and $U_o(\bar{x}_2, R, \epsilon') \subseteq U_o(\bar{x}_1, R, \epsilon)$. Then, by conicality, $U_o(\bar{x}_2, R, \epsilon') \cap \partial_\sigma X \supseteq U_o(\bar{x}_2, RC/\epsilon', C) \cap \partial_\sigma X = B_{d_{o,C}}(\bar{x}_2, \epsilon'/RC)$.

(iii) Let $\mu = \min(C, C')$ and $M = \max(C, C')$. By conicality of σ, for any $\bar{x}_1, \bar{x}_2 \in \partial X$ we have $d_{o,M}(\bar{x}_1, \bar{x}_2) \leq d_{o,\mu}(\bar{x}_1, \bar{x}_2) \leq (M/\mu) d_{o,M}(\bar{x}_1, \bar{x}_2)$. The claim follows.

(iv) Since quasisymmetries are closed under composition, in view of (iii), we can assume that $C > 2d(o, o')$. Let $\bar{x}_1, \bar{x}_2 \in \partial X$ be different and $t := d_{o,C}(\bar{x}_1, \bar{x}_2)^{-1}$. If $d(\varrho_{o',\bar{x}_1}(t), \varrho_{o',\bar{x}_2}(t)) \geq C$, then $d_{o',C}(\bar{x}_1, \bar{x}_2) \geq t^{-1} = d_{o,C}(\bar{x}_1, \bar{x}_2)$. Otherwise, observe that by the triangle inequality and Proposition 2.1(i),

$$
\begin{aligned}
d(\varrho_{o',\bar{x}_1}(t), \varrho_{o',\bar{x}_2}(t)) &\geq d(\varrho_{o,\bar{x}_1}(t), \varrho_{o,\bar{x}_2}(t)) \\
&\quad - d(\varrho_{o,\bar{x}_1}(t), \varrho_{o',\bar{x}_1}(t)) - d(\varrho_{o,\bar{x}_2}(t), \varrho_{o',\bar{x}_2}(t)) \geq C - 2d(o, o').
\end{aligned}
$$

By conicality of σ,

$$
d\left(\varrho_{o',\bar{x}_1}\left(\frac{tC}{C - 2d(o,o')}\right), \varrho_{o',\bar{x}_2}\left(\frac{tC}{C - 2d(o,o')}\right)\right) \geq \frac{C}{C - 2d(o,o')} d(\varrho_{o',\bar{x}_1}(t), \varrho_{o',\bar{x}_2}(t)) \geq C,
$$

therefore $d_{o',C}(\bar{x}_1, \bar{x}_2)^{-1} \leq tC/(C - 2d(o, o'))$.

Summarising both cases, $d_{o,C}(\bar{x}_1, \bar{x}_2) \leq (C/(C - 2d(o, o')))d_{o',C}(\bar{x}_1, \bar{x}_2)$. Therefore, as we can swap o with o' in the above reasoning, the claim follows.

(v) Observe that the action of each element $g \in G$ on X induces an an isometry between $(\partial_\sigma X, d_{o,C})$ and $(\partial_\sigma X, d_{go,C})$, therefore the claim follows by (iv) and the fact that quasisymmetries are closed under composition. □

7. Axes, flats, and the topology of boundary

We begin this section with a brief recap of some standard terminology. For an isometry φ of a metric space (X, d), we define $|\varphi| := \inf_{x \in X} d(x, \varphi(x))$, $\mathrm{Min}(\varphi) := \{x \in X : d(x, \varphi(x)) = |\varphi|\}$, and we call an isometric embedding $\gamma \colon \mathbb{R} \to X$ an *axis* of φ if there exists a number $T > 0$ such that $\varphi(\gamma(t)) = \gamma(t + T)$ for all $t \in \mathbb{R}$. By the triangle inequality, for any $x \in X$ and $n \in \mathbb{N}$ we have that

$$nT = d(\gamma(0), \varphi^n(\gamma(0))) \leq d(\gamma(0), x) + d(x, \varphi^n(x)) + d(\varphi^n(x), \varphi^n(\gamma(0)))$$
$$\leq 2d(\gamma(0), x) + nd(x, \varphi(x)),$$

therefore $T = |\varphi|$, and $\operatorname{im}\gamma \subseteq \operatorname{Min}(\varphi)$. For a bicombing σ on X, we call an axis γ of φ a *σ-axis* if $\operatorname{im}\gamma|_{[s,t]} = \operatorname{im}\sigma_{\gamma(s),\gamma(t)}$ for all real numbers $s < t$.

Proposition 7.1 (Proposition VII). *Let $G = G_1 *_Z G_2$ (with $G_1 \neq Z \neq G_2$), where Z is virtually $\mathbb{Z}$, act geometrically on a proper metric space (X, d_X) that admits a* $\mathfrak{ccc}$*, reversible, G-equivariant bicombing σ. Then there exists a separating pair of points in the boundary $\partial_\sigma X$.*

Proof. We briefly recall the normal form for the amalgamated product as in e.g. [Ser80, Theorem 1 in I.1.2], which we use in this proof. For $i = 1, 2$, choose a set R_i of representatives of non-trivial right cosets of Z in G_i. Then each element $g \in G$ may be represented in a unique way as $g = z \cdot r_1 \cdot r_2 \cdot \ldots \cdot r_k$, where $z \in Z$, $r_j \in R_1 \cup R_2$ for $1 \leq j \leq k$, and $r_j \in R_1$ iff $r_{j+1} \in R_2$ for $1 \leq j \leq k-1$. We refer to k as the length of the normal form representation of g, and for $1 \leq j \leq k$ we refer to r_j as the j-th term of this representation.

Since G acts geometrically on a proper metric space, it is finitely generated. Let Γ be the Cayley graph for G over a finite set of generators S contained in $G_1 \cup G_2$. Let

$$A_i := \{g \in G : \text{the first term in the normal form representation for } g \text{ belongs to } G_i\}$$

for $i = 1, 2$. Then G is the disjoint union of A_1, Z and A_2. Since the generating set S is a subset of $G_1 \cup G_2$, the lengths of the normal form representations for any two group elements connected by an edge in the graph Γ differ by at most 1, and, for $i = 1, 2$, the neighbours in the graph Γ of the elements of A_i are contained in the set $A_i \cup Z$, which implies that any path in Γ from an element of A_1 to an element of A_2 must pass through the set Z.

Let M be a cyclic subgroup of finite index of Z, and let m be a generator of M. By [DL16, Proposition 5.5], m has a σ-axis μ. Let α be the quasi-isometry given by $G \ni g \mapsto g\mu(0) \in X$. Let $C > 0$ be such that M is C-dense in Z (with respect to the metric d_Γ from the graph Γ), the image $\alpha(G)$ is C-dense in X, and for all $g, g' \in G$ the inequality $C^{-1}d_\Gamma(g, g') - C \leq d_X(\alpha(g), \alpha(g')) \leq Cd_\Gamma(g, g') + C$ holds. Put $X_i := B(\alpha(A_i \cup Z), C+1)$ for $i = 1, 2$. Clearly, X_1 and X_2 are open subsets of X such that $X_1 \cup X_2 = X$. Consider any element $x \in X_1 \cap X_2$. Then there exist $a_i \in A_i \cup Z$, where $i = 1, 2$, such that $d_X(\alpha(a_i), x) < C + 1$. Therefore $d_X(\alpha(a_1), \alpha(a_2)) < 2C + 2$, so $d_\Gamma(a_1, a_2) < 3C^2 + 2C$. Since, as discussed above, the path in Γ from a_1 to a_2 necessarily passes through Z, we have that $d_\Gamma(a_i, Z) < 3C^2 + 2C$ for $i = 1, 2$, which, as $\alpha(M) \subseteq \operatorname{im}\mu$, gives that

$$\begin{aligned} d_X(x, \operatorname{im}\mu) \le d_X(x, \alpha(M)) \le d_X(x, \alpha(a_1)) + d_X(\alpha(a_1), \alpha(M)) \\ < C + 1 + C d_\Gamma(a_1, M) + C \le 2C + 1 + C(d_\Gamma(a_1, Z) + C) \\ < 2C + 1 + C(3C^2 + 2C + C) =: \mathbf{C}. \end{aligned}$$

Therefore $X_1 \cap X_2 \subseteq B(\operatorname{im}\mu, \mathbf{C})$.

Let $\xi_+, \xi_- \colon [0, \infty) \to X$ be the σ-rays defined by $\xi_+(t) := \mu(t)$ and $\xi_-(t) := \mu(-t)$. We show that the pair of points $[\xi_+], [\xi_-]$ disconnects the boundary $\partial_\sigma X$. Consider any σ-ray $\zeta \notin \{\xi_+, \xi_-\}$ based in $\mu(0)$. By Proposition 2.1(iii), there exists $r > 0$ such that $d(\zeta(t), \operatorname{im}\mu) \ge \mathbf{C} + 2$ for any $t \ge r$, in particular there exists unique $i \in \{1, 2\}$ such that $\zeta(t) \in X_i$ for all $t \ge r$. Denote

$$E_i := \{[\zeta] : \zeta \text{ is a } \sigma\text{-ray}, \zeta(0) = \mu(0), (\exists r > 0)(\forall t \ge r)(\zeta(t) \in X_i)\};$$

we have that $\partial_\sigma X$ is the disjoint union of E_1, E_2 and $\{[\xi_-], [\xi_+]\}$. It remains to show that both E_1 and E_2 are non-empty open subsets of the boundary $\partial_\sigma X$. First, we prove the openness. Let ζ, r and i be as above. Consider any σ-ray η such that $d(\eta(r), \zeta(r)) < 1$. By conicality and the triangle inequality, $d(\eta(t), \operatorname{im}\mu) \ge d(\eta(r), \operatorname{im}\mu) \ge \mathbf{C} + 1$ for all $t \ge r$, therefore $\eta(t) \in X_i$, and $[\eta] \in E_i$. Now we prove that both sets E_i are non-empty. Below we prove that $E_1 \neq \emptyset$, the case of E_2 is symmetric. Let $g_1 \in R_1$ and $g_2 \in R_2$, and let $g := g_1 g_2$. Then for all $n \in \mathbb{N} \setminus \{0\}$, since the normal form for g^n is the n-fold concatenation of the normal form for g, we have that $g^n \in A_1$, and, by the discussion in the second paragraph of this proof, $d_\Gamma(g^n, Z) \ge 2n$. Let γ be a σ-axis for g in X, see [DL15, Proposition 5.5], and define the σ-ray ζ to be the σ-ray originating in $\mu(0)$ asymptotic to $\gamma|_{[0,\infty)}$. Proposition 2.1(i) and the g-invariance of the metric d_X give that

$$\begin{aligned} d_X(\zeta(|g|n), \alpha(g^n)) \le d_X(\zeta(|g|n), \gamma(|g|n)) + d_X(\gamma(|g|n), \alpha(g^n)) \\ \le d_X(\zeta(0), \gamma(0)) + d_X(g^n\gamma(0), g^n\mu(0)) = 2d_X(\gamma(0), \mu(0)) =: D. \end{aligned}$$

We also have the following chain of inequalities

$$\begin{aligned} d_X(\alpha(g^n), \operatorname{im}\mu) \ge d_X(\alpha(g^n), \alpha(M)) - |m| \\ \ge d_X(\alpha(g^n), \alpha(Z)) - |m| \ge C^{-1} d_\Gamma(g^n, Z) - C - |m| \ge C^{-1} \cdot 2n - C - |m|. \end{aligned}$$

Jointly, the last two chains of inequalities have the following two consequences, which together give that $[\zeta]$ belongs to E_1. First, for any $n \in \mathbb{N}$, if $\zeta(|g|n)$ belongs to X_2, then openness of X_1 and X_2 gives that the σ-geodesic from the point $\zeta(|g|n)$ to the point $\alpha(g^n)$, belonging to X_1, passes through $X_1 \cap X_2$; therefore

$$D \ge d(\alpha(g^n), \zeta(|g|n)) \ge d_X(\alpha(g^n), X_1 \cap X_2) \ge d_X(\alpha(g^n), \operatorname{im}\mu) - \mathbf{C},$$

which tends to ∞ when $n \to \infty$; therefore $\zeta(|g|n)$ belongs to X_2 only for finitely many $n \in \mathbb{N}$, so $[\zeta]$ does not belong to E_2. Second, the inequality $d_X(\zeta(|g|n), \operatorname{im}\mu) \geq 1$ holds for sufficiently large n, therefore ζ is not asymptotic to any of the σ-rays ξ_-, ξ_+. $\square$

Proposition 7.2 (Proposition VIII). *Let G be group that contains a free abelian subgroup $\mathbb{Z}^n \cong A < G$ and acts geometrically on a proper metric space X that admits a* $\mathfrak{ccc}$*, reversible, G-equivariant bicombing σ^X. Then $\partial_{\sigma^X} X$ contains a homeomorphic copy of S^{n-1}. Moreover, if A is of finite index in G, then $\partial_{\sigma^X} X \cong S^{n-1}$.*

Proof. By [DL16, Theorem 1.2] X contains an isometric copy F of an n-dimensional normed space on which A acts geometrically by translations. Observe that F admits a (unique) $\mathfrak{ccc}$, A-equivariant bicombing σ^F, which consists of linear segments (recall [DL15, Theorem 3.3]). Note that in general F is not σ^X-convex in X (see [DL16, Example 6.3]) — if it was so, then the assertion would easily follow, as we would have $\sigma^F = \sigma^X|_{F\times F\times[0,1]}$, which would allow us to view $S^{n-1} \cong \partial_{\sigma^F} F$ as a subset of $\partial_{\sigma^X} X$. We shall define a homeomorphic embedding $\Phi\colon \partial_{\sigma^F} F \to \partial_{\sigma^X} X$. If, additionally, A is of finite index in G, then Φ turns out to be a surjection.

Fix a basepoint $o \in F$. For each $a \in A$, denote by ξ_a^F the σ^F-ray that originates in o and contains ao, pick a σ^X-axis γ_a^X in X (see [DL16, Proposition 5.5]), and put $\xi_a^X := \gamma_a^X|_{[0,\infty)}$.

The construction of the map Φ, which consists in continuously extending the map induced by sending ξ_a^F to ξ_a^X for each $a \in A$, is presented in detail below the following claim, which is used to justify the correctness of the construction and various properties of Φ. Picking one σ^X-axis γ_a^X for each $a \in A$, rather than considering the set of all such σ^X-axes, is more of an editorial choice; in particular, the defined map Φ does not depend on this choice, as may be seen using part (A) of the claim below. The colours are used in the further text to highlight the key places of the formulas and to aid the presentation of the flow of the argument.

Claim. *Let $a \in A$ and γ_1, γ_2 be axes (which are not necessarily σ^X-axes) of a. Then*

(A) $d(\gamma_1(t), \gamma_2(t)) \leq 2|a| + d(\gamma_1(0), \gamma_2(0)) =: C(|a|, \gamma_1(0), \gamma_2(0))$ *for any $t \in \mathbb{R}$.*

Let $a_1, a_2 \in A$ and $r > 0$. Then

(B) $$\left|d(\varrho^X_{o,[\xi^X_{a_1}]}(r), \varrho^X_{o,[\xi^X_{a_2}]}(r)) - d(\xi^F_{a_1}(r), \xi^F_{a_2}(r))\right| \leq C'(a_1) + C'(a_2),$$
$$\text{where } C'(a) = C(|a|, \xi^X_a(0), \xi^F_a(0)) + d(\xi^X_a(0), o);$$

(B♯) $$d(\varrho^X_{o,[\xi^X_{a_1}]}(r), \varrho^X_{o,[\xi^X_{a_2}]}(r)) \leq d(\xi^F_{a_1}(r), \xi^F_{a_2}(r)).$$

Proof. (A) For $0 \le t \le |a|$ the claim follows by the triangle inequality:

$$d(\gamma_1(t), \gamma_2(t)) \le d(\gamma_1(t), \gamma_1(0)) + d(\gamma_1(0), \gamma_2(0)) + d(\gamma_2(0), \gamma_2(t)) \\ \le |a| + d(\gamma_1(0), \gamma_2(0)) + |a|.$$

The case of arbitrary $t \in \mathbb{R}$ follows from a-invariance of the metric d and shifting along axes.

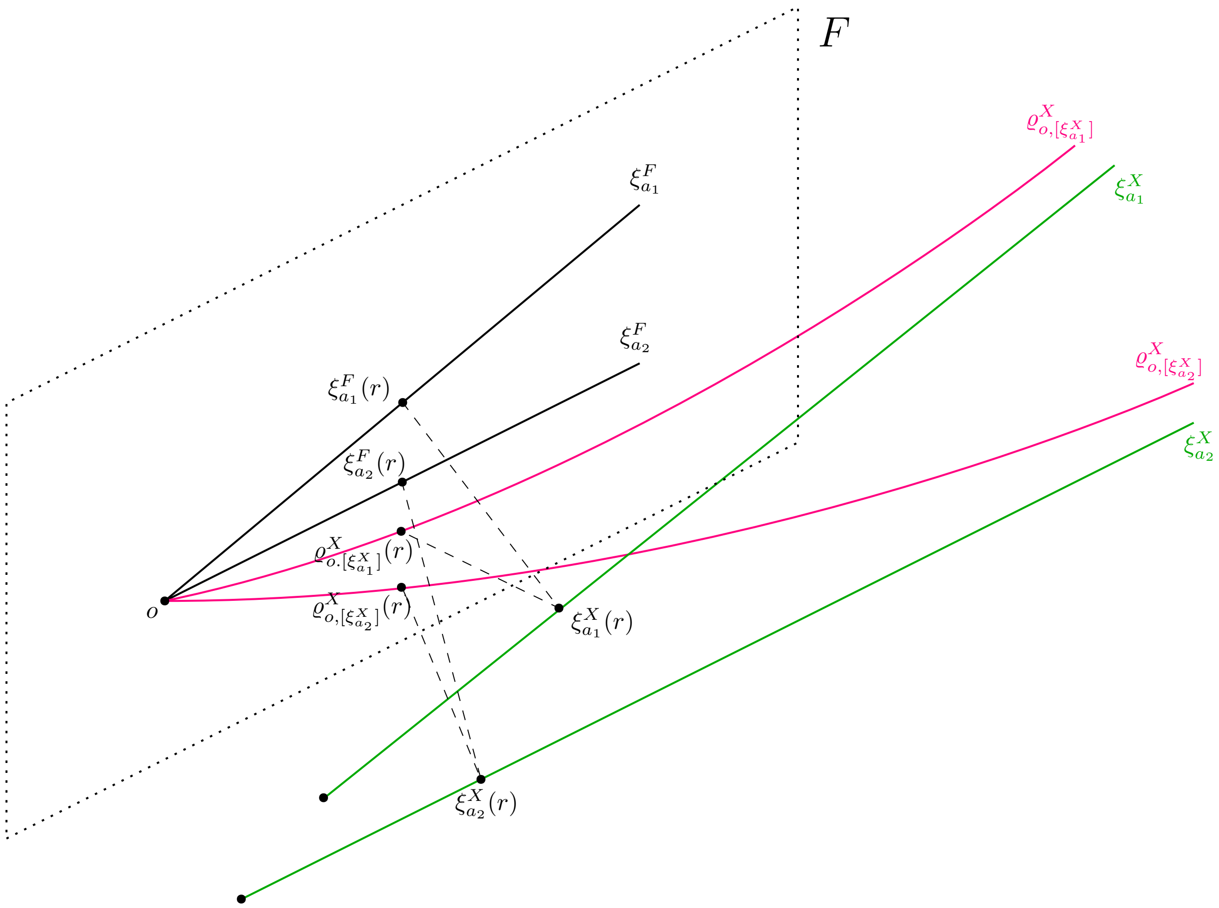


Figure 3: Claim (B).

(B) By the triangle inequality, we obtain that

$$\left| d(\varrho^X_{o,[\xi^X_{a_1}]}(r), \varrho^X_{o,[\xi^X_{a_2}]}(r)) - d(\xi^F_{a_1}(r), \xi^F_{a_2}(r)) \right| \\ \le d(\varrho^X_{o,[\xi^X_{a_1}]}(r), \xi^F_{a_1}(r)) + d(\varrho^X_{o,[\xi^X_{a_2}]}(r), \xi^F_{a_2}(r)) \\ \le d(\varrho^X_{o,[\xi^X_{a_1}]}(r), \xi^X_{a_1}(r)) + d(\xi^X_{a_1}(r), \xi^F_{a_1}(r)) + d(\varrho^X_{o,[\xi^X_{a_2}]}(r), \xi^X_{a_2}(r)) + d(\xi^X_{a_2}(r), \xi^F_{a_2}(r)).$$

The claim follows, as for any $i \in \{1, 2\}$ the following two inequalities hold: by Proposition 2.1(i), we have that $d(\varrho^X_{o,[\xi^X_{a_i}]}(r), \xi^X_{a_i}(r)) \le d(\varrho^X_{o,[\xi^X_{a_i}]}(0), \xi^X_{a_i}(0))$; and, by Claim (A) applied to axes containing $\xi^X_{a_i}$ and $\xi^F_{a_i}$, we have that $d(\xi^X_{a_i}(r), \xi^F_{a_i}(r)) \le C(|a_i|, \xi^X_{a_i}(0), \xi^F_{a_i}(0))$.

(B$\sharp$) Fix $R > r$. By scaling in F, $d(\xi^F_{a_1}(R), \xi^F_{a_2}(R)) = d(\xi^F_{a_1}(r), \xi^F_{a_2}(r)) \cdot R/r$. Therefore, by conicality of σ^X and (B) (applied for R and a_1, a_2),

$$\begin{aligned}d(\varrho^X_{o,[\xi^X_{a_1}]}(r), \varrho^X_{o,[\xi^X_{a_2}]}(r)) &\leq d(\varrho^X_{o,[\xi^X_{a_1}]}(R), \varrho^X_{o,[\xi^X_{a_2}]}(R)) \cdot r/R \\ &\leq (d(\xi^F_{a_1}(R), \xi^F_{a_2}(R)) + C'(a_1) + C'(a_2)) \cdot r/R \\ &= d(\xi^F_{a_1}(r), \xi^F_{a_2}(r)) + (C'(a_1) + C'(a_2)) \cdot r/R.\end{aligned}$$

The claim follows, as R may be chosen arbitrarily large.

Let $F_A \subseteq \partial_{\sigma^F} F$ be the asymptotic classes of σ^F-rays belonging the set $\{\xi^F_a : a \in A\}$. Define $\Phi_A \colon F_A \to \partial_{\sigma^X} X$ to be the map induced by the map $\xi^F_a \mapsto \xi^X_a$. Claim (B$\sharp$) implies that the map Φ_A is well-defined (i.e. if two elements $a, b \in A$ are such that $\xi^F_a = \xi^F_b$, then ξ^X_a and ξ^X_b are asymptotic) and 1-Lipschitz from $(F_A, d^F_{o,1})$ to $(\partial_{\sigma^X} X, d^X_{o,1})$, where $d^F_{o,1}$ and $d^X_{o,1}$ are the metrics discussed in Section 6. Therefore, since F_A is dense in $\partial_{\sigma^F} F$, the map Φ_A can be extended continuously to a map $\Phi \colon \partial_{\sigma^F} F \to \partial_{\sigma^X} X$.

We show that Φ is one-to-one. Since $\partial_{\sigma^F} F \cong S^{n-1}$ is compact, this will imply that Φ is a homeomorphic embedding onto its image. Let $\xi^F_1 \neq \xi^F_2$ be σ^F-rays based in o, and let $a^i_n \in A$ for $i = 1, 2$ and $n \in \mathbb{N}$ be such that $[\xi^F_{a^i_n}]$ converges to $[\xi^F_i]$ in $\partial_{\sigma^F} F$ for $i = 1, 2$. We may choose $r > 0$ such that $d(\xi^F_1(r), \xi^F_2(r)) \geq 8$, and $N \in \mathbb{N}$ such that for any $n \geq N$ and $i = 1, 2$ we have $d(\xi^F_{a^i_n}(r), \xi^F_i(r)) \leq 1$. In particular, $d(\xi^F_{a^1_N}(r), \xi^F_{a^2_N}(r)) \geq 6$. By Claim (B) and scaling in F, for any $R > r$,

$$\begin{aligned}d(\varrho^X_{o,[\xi^X_{a^1_N}]}(R), \varrho^X_{o,[\xi^X_{a^2_N}]}(R)) &\geq d(\xi^F_{a^1_N}(R), \xi^F_{a^2_N}(R)) - C'(a^1_N) - C'(a^2_N) \\ &= d(\xi^F_{a^1_N}(r), \xi^F_{a^2_N}(r)) \cdot R/r - C'(a^1_N) - C'(a^2_N) \geq 6R/r - C'(a^1_N) - C'(a^2_N),\end{aligned}$$

which is greater than $5R/r$ for sufficiently large R. Using Claim (B$\sharp$) and scaling in F, we obtain that for any $n \geq N$

$$\begin{aligned}d(\varrho^X_{o,[\xi^X_{a^i_n}]}(R), \varrho^X_{o,[\xi^X_{a^i_N}]}(R)) &\leq d(\xi^F_{a^i_n}(R), \xi^F_{a^i_N}(R)) \\ &\leq d(\xi^F_{a^i_n}(R), \xi^F_i(R)) + d(\xi^F_i(R), \xi^F_{a^i_N}(R)) \leq R/r + R/r = 2R/r.\end{aligned}$$

Therefore

$$\begin{aligned}&d(\varrho^X_{o,[\xi^X_{a^1_n}]}(R), \varrho^X_{o,[\xi^X_{a^2_m}]}(R)) \\ &\quad\geq d(\varrho^X_{o,[\xi^X_{a^1_N}]}(R), \varrho^X_{o,[\xi^X_{a^2_N}]}(R)) - d(\varrho^X_{o,[\xi^X_{a^1_n}]}(R), \varrho^X_{o,[\xi^X_{a^1_N}]}(R)) - d(\varrho^X_{o,[\xi^X_{a^2_m}]}(R), \varrho^X_{o,[\xi^X_{a^2_N}]}(R)) \\ &\quad\geq 5R/r - 2R/r - 2R/r \geq R/r\end{aligned}$$

for any $n, m \geq N$ and sufficiently large R. Therefore, passing to the limit with n and m, one obtains that $d(\varrho^X_{o,\Phi([\xi^F_1])}(R), \varrho^X_{o,\Phi([\xi^F_2])}(R)) \geq R/r > 0$, which implies that $\Phi([\xi^F_1]) \neq \Phi([\xi^F_2])$.

Now we prove the last part of the statement of this proposition: assume additionally that A is of finite index in G; we show that then Φ is onto. Let $C > 0$ be such that the set Ao of A-translates of o satisfies $B(Ao, C) = X$. Let ξ^X be a σ^X-ray based in o. For every $n \in \mathbb{N}$ there exists $a_n \in A$ such that $d(a_n o, \xi^X(n)) \leq C$. It suffices to show that $\varrho^X_{o,[\xi^X_{a_n}]}(r)$ converges to $\xi^X(r)$ when $n \to \infty$ for any $r > 0$. First, observe that $||a_n| - n| = |d(a_n o, o) - d(o, \xi^X(n))| \leq d(a_n o, \xi^X(n)) \leq C$, therefore, in particular, $|a_n| \to \infty$. Second, since translating a σ^X-axis of $a \in A$ by an element of the centraliser $C(a) \supseteq A$ produces a σ^X-axis of a, and $B(Ao, C) = X$, there exists a σ^X-axis θ^X_a of a such that $d(\theta^X_a(0), o) \leq C$; put $\zeta^X_a := \theta^X_a|_{[0,\infty)}$; by Claim (A), the σ^X-rays ζ^X_a and ξ^X_a are asymptotic. By the triangle inequality and Proposition 2.1(i), we have that

$$
\begin{aligned}
&d(\varrho^X_{o,[\xi^X_{a_n}]}(|a_n|), \xi^X(|a_n|)) = d(\varrho^X_{o,[\zeta^X_{a_n}]}(|a_n|), \xi^X(|a_n|)) \\
&\quad \leq d(\varrho^X_{o,[\zeta^X_{a_n}]}(|a_n|), \zeta^X_{a_n}(|a_n|)) + d(\zeta^X_{a_n}(|a_n|), a_n o) + d(a_n o, \xi^X(n)) + d(\xi^X(n), \xi^X(|a_n|)) \\
&\quad\quad \leq d(\varrho^X_{o,[\zeta^X_{a_n}]}(0), \zeta^X_{a_n}(0)) + d(a_n \zeta^X_{a_n}(0), a_n o) + C + ||a_n| - n| \leq C + C + C + C = 4C;
\end{aligned}
$$

then the conicality of σ^X gives that $d(\varrho^X_{o,[\xi^X_{a_n}]}(r), \xi^X(r)) \leq 4Cr/|a_n|$ whenever $r \leq |a_n|$. Therefore, since $|a_n| \to \infty$, we have that $d(\varrho^X_{o,[\xi^X_{a_n}]}(r), \xi^X(r)) \to 0$ for all $r \geq 0$. $\square$

8. Almost geodesic completeness

A space X that admits a bicombing σ is *almost σ-geodesically complete* if for some (equivalently, for all — see Proposition 2.1(i)) basepoint $o \in X$ there exists a universal constant $C > 0$ such that for each point $x \in X$ there is a σ-ray ξ such that $\xi(0) = o$ and $\operatorname{im} \xi \cap \overline{B}(x, C) \neq \emptyset$. In this section we prove the following theorem.

Theorem 8.1 (Theorem X; cf. [GO07, Corollary 3]). *Assume that X is a proper non-compact finite-dimensional geodesic metric space that admits a $\mathfrak{ccc}$ geodesic bicombing σ and a cocompact group action via isometries. Then X is almost σ-geodesically complete.*

The proof from [Ont05; GO07], where the space X is assumed to be CAT(0), can be translated to the context of spaces admitting a $\mathfrak{ccc}$ geodesic bicombing. We expand on it below.

8.1. Preparatory lemmas

Recall Definition 2.3 and Proposition 2.4. Let $\operatorname{Cone}_o(A) := \exp_o(A \times [0, \infty])$ for any set $A \subseteq \partial_\sigma X$ and basepoint $o \in X$.

Lemma 8.2 (cf. proof of [GO07, Main Theorem]). *Let X be a proper metric space that admits a* ccc *geodesic bicombing. Then for every non-empty closed set $A \subseteq \partial_\sigma X$ there exists a closed set $D \subseteq \overline{X}_\sigma$ such that (i) $D \cap \partial_\sigma X = A$, (ii) $\mathrm{Cone}_o(A) \subseteq D$, (iii) $D \cap X \subseteq \overline{B}_X(\mathrm{Cone}_o(A) \cap X, 1)$, and (iv) D is a strong deformation retract of $\overline{X}_\sigma$.*

Proof. For any $\bar{a} \in A$ and $\bar{x} \in \overline{X}_\sigma$, we have by convexity of σ that the function $\delta^{\bar{a}}_{\bar{x}} \colon [0, \infty) \cap [0, \ell_o(\bar{x})] \to \mathbb{R}$ given by $\delta^{\bar{a}}_{\bar{x}}(s) = d(\exp_o(\bar{a}, s), \exp_o(\bar{x}, s))$ is non-decreasing, strictly increasing on $(\delta^{\bar{a}}_{\bar{x}})^{-1}((0, \infty))$ and continuous. Since A is compact, the function $\mu^A_{\bar{x}} \colon [0, \infty) \cap [0, \ell_o(\bar{x})] \to \mathbb{R}$ given by $\mu^A_{\bar{x}}(s) = \min_{\bar{a} \in A} \delta^{\bar{a}}_{\bar{x}}(s)$ is well-defined, and is non-decreasing and strictly increasing on $(\mu^A_{\bar{x}})^{-1}((0, \infty))$, as the functions $\delta^{\bar{a}}_{\bar{x}}$ are; it is continuous by the following argument. Let $s \in [0, \infty) \cap [0, \ell_o(\bar{x})]$. Consider any sequence $(s_n) \subseteq [0, \infty) \cap [0, \ell_o(\bar{x})]$ converging to s from above. Let $\bar{a} \in A$ be such that $\mu^A_{\bar{x}}(s) = \delta^{\bar{a}}_{\bar{x}}(s)$. Then we have $\delta^{\bar{a}}_{\bar{x}}(s_n) \geq \mu^A_{\bar{x}}(s_n) \geq \mu^A_{\bar{x}}(s) = \delta^{\bar{a}}_{\bar{x}}(s)$. As the left hand side converges to the right hand side as $n \to \infty$, we have that $\mu^A_{\bar{x}}(s_n) \to \mu^A_{\bar{x}}(s)$. Now, assume that we have a sequence $(s_n) \subseteq [0, \infty) \cap [0, \ell_o(\bar{x})]$ approaching s from below. Let $\bar{a}_n$ be such that $\mu^A_{\bar{x}}(s_n) = \delta^{\bar{a}_n}_{\bar{x}}(s_n)$. By compactness of A, each subsequence of $(n)_{n \in \mathbb{N}}$ admits a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $\bar{a}_{n_k}$ is convergent to some $\bar{a} \in A$. We have the following inequalities: $\mu^A_{\bar{x}}(s) \geq \mu^A_{\bar{x}}(s_{n_k}) = d(\exp_o(\bar{x}, s_{n_k}), \exp_o(\bar{a}_{n_k}, s_{n_k}))$. By continuity of $\exp_o$, passing to the limit with k we obtain that $\mu^A_{\bar{x}}(s) \geq \mu^A_{\bar{x}}(s_{n_k}) \to \delta^{\bar{a}}_{\bar{x}}(s) \geq \mu^A_{\bar{x}}(s)$. Therefore $\mu^A_{\bar{x}}(s_n) \to \mu^A_{\bar{x}}(s)$.

Define $\omega \colon \overline{X}_\sigma \to [0, \infty]$ by $\omega(\bar{x}) = \sup\{s \in [0, \infty) \cap [0, \ell_o(\bar{x})] : \mu^A_{\bar{x}}(s) \leq 1\}$ (in slightly informal terms: 'walk from o along the σ-geodesic/ray to $\bar{x}$ until reaching $\bar{x}$ or diverging to a "sphere-wise" distance at least 1 from all of the σ-rays that begin in o and end in A; the distance covered is $\omega(\bar{x})$') and let $D := \{\bar{x} \in \overline{X}_\sigma : \ell_o(\bar{x}) = \omega(\bar{x})\}$. It easily follows that D satisfies (ii) and (iii), and that $A \subseteq D \cap \partial_\sigma X$. The other inclusion required by property (i) is satisfied, as

$$\begin{aligned} D \cap \partial_\sigma X &= \{\bar{x} \in \partial_\sigma X : (\forall s \geq 0)(\exists \bar{a} \in A)(d(\varrho_{o,\bar{x}}(s), \varrho_{o,\bar{a}}(s)) \leq 1)\} \\ &\subseteq \{\bar{x} \in \partial_\sigma X : (\forall s \geq 0)(\exists \bar{a} \in A)(d_{o,2}(\bar{x}, \bar{a}) \leq s^{-1})\} \subseteq \overline{A} = A \end{aligned}$$

(recall Proposition 6.1(i)). It is sufficient to prove that ω is continuous, as then it immediately follows that D is closed, and that property (iv) is satisfied, as then we have the following retraction: $\overline{X}_\sigma \times [0, \infty] \ni (\bar{x}, s) \mapsto \exp_o(\bar{x}, \max(s, \omega(\bar{x}))) \in \overline{X}_\sigma$.

Claim. *Assume that $(\bar{x}_n) \subseteq \overline{X}_\sigma$ converges to $\bar{x} \in \overline{X}_\sigma$ and $\omega(\bar{x}_n)$ converges to some $t \in [0, \infty]$. Then* (A) *if $\ell_o(\bar{x}) > s > t$, then $\mu^A_{\bar{x}}(s) \geq 1$;* (B) *if $t > s \geq 0$ then $\mu^A_{\bar{x}}(s) \leq 1$;* (C) $\omega(\bar{x}) = t$.

Proof. (A) If not, then there exists $\bar{a} \in A$ such that $d(\exp_o(\bar{x}, s), \exp_o(\bar{a}, s)) = \delta^{\bar{a}}_{\bar{x}}(s) < 1$. Since $\bar{x}_n \to \bar{x}$, for large enough n we have that $\delta^{\bar{a}}_{\bar{x}_n}(s) = d(\exp_o(\bar{x}_n, s), \exp_o(\bar{a}, s)) < 1$, and, as ℓ_o is continuous, $s < \ell_o(\bar{x}_n)$. This implies that $\omega(\bar{x}_n) > s$ for sufficiently large n, thus $t \geq s$. Contradiction.

(B) Let $\bar{a}_n \in A$ be such that $\delta_{\bar{x}_n}^{\bar{a}_n}(\omega(\bar{x}_n)) = \mu_{\bar{x}_n}^A(\omega(\bar{x}_n))$. Then we have that $1 \geq \delta_{\bar{x}_n}^{\bar{a}_n}(\omega(\bar{x}_n)) = d(\exp_o(\bar{x}_n, \omega(\bar{x}_n)), \exp_o(\bar{a}_n, \omega(\bar{x}_n)))$. Since $\omega(\bar{x}_n) \to t$, for large enough n we have that $\omega(\bar{x}_n) \geq s$, therefore $1 \geq d(\exp_o(\bar{x}_n, s), \exp_o(\bar{a}_n, s))$. Therefore, by compactness of A, there exists $\bar{a} \in A$ such that $1 \geq d(\exp_o(\bar{x}, s), \exp_o(\bar{a}, s)) = \delta_{\bar{x}}^{\bar{a}}(s) \geq \mu_{\bar{x}}^A(s)$.

(C) First, note that passing to the limit with $\omega(\bar{x}_n) \leq \ell_o(\bar{x}_n)$ gives that $t \leq \ell_o(\bar{x})$. If $t = \ell_o(\bar{x}) > 0$, then by (B) we have that $\omega(\bar{x}) \geq s$ for any $s < t$, therefore $t = \ell_o(\bar{x}) \geq \omega(\bar{x}) \geq t$, so $\omega(\bar{x}) = t$. If $t = \ell_o(\bar{x}) = 0$, then $\omega(\bar{x}) = 0 = t$, as $0 \leq \omega(\bar{x}) \leq \ell_o(\bar{x}) = 0$. Otherwise, we have that $t < \ell_o(\bar{x})$. Claim (A) implies that $\mu_{\bar{x}}^A(t) \geq 1$. Since $\ell_o(\bar{x}_n) \to \ell_o(\bar{x})$, we have for sufficiently large n that $\omega(\bar{x}_n) < \ell_o(\bar{x}_n)$; also note that $\omega(\bar{x}_n) \geq 1/2$, which in the limit implies that $t \geq 1/2 > 0$, since $\mu_{\bar{x}_n}^A(\omega(\bar{x}_n)) = 1$ and the diameter of $B(o, 1/2)$ is not greater than 1; therefore Claim (B) implies that $\mu_{\bar{x}}^A(t) \leq 1$. Since $\mu_{\bar{x}}^A$ is increasing on $(\mu_{\bar{x}}^A)^{-1}((0,\infty))$, t is the unique number such that $\mu_{\bar{x}}^A(t) = 1$, and $\omega(\bar{x}) = t$.

Continuity of ω now follows, as each subsequence of a convergent sequence in $\overline{X}_\sigma$ admits a subsequence that satisfies the assumptions of Claim (C) above. $\square$

For a simplicial complex K, below we consider it with the piecewise–unit-ℓ^∞ metric. That is, we endow it with the gluing metric arising from the identification of each k-simplex $[v_0, \ldots, v_k]$ of K with the subspace $\{(\lambda_0, \ldots, \lambda_k) : \lambda_0, \ldots, \lambda_k \geq 0 \text{ and } \lambda_0 + \ldots + \lambda_k = 1\}$ of $\mathbb{R}^{k+1}$ with the supremum metric.

Two spaces X, Y have the same *bounded homotopy type* if there exist (continuous) maps $f\colon X \to Y$ and $g\colon Y \to X$, and *bounded homotopies* $\mathfrak{h}^X\colon X \times [0,1] \to X$ between $g \circ f$ and id_X, and $\mathfrak{h}^Y\colon Y \times [0,1] \to Y$ between $f \circ g$ and id_Y, i.e. homotopies such that the diameters of the trajectories $\mathfrak{h}^X(x, \cdot\,)$ for $x \in X$ and $\mathfrak{h}^Y(y, \cdot\,)$ for $y \in Y$ are bounded by a constant independent of the choice of x and y. The maps f and g above are called *bounded homotopy equivalences*.

Lemma 8.3 (cf. [BH99, Lemma I.7A.15]). *Let X be a metric space that admits a locally finite open cover $\boldsymbol{\mathcal{U}} = \{B(x_i, \epsilon) : i \in I\}$ for some $\epsilon > 0$. Assume that for $k \in \{1, 3\}$ each ball $B(x_i, k\epsilon)$ admits a continuous function $\sigma^{i,k\epsilon}\colon B(x_i, k\epsilon) \times B(x_i, k\epsilon) \times [0,1] \to B(x_i, k\epsilon)$ satisfying $\sigma^{i,k\epsilon}(x, x', 0) = x$, $\sigma^{i,k\epsilon}(x, x', 1) = x'$ for all $x, x' \in B(x_i, k\epsilon)$, and that these functions are such that $\sigma^{i,k\epsilon}|_{(B(x_i,k\epsilon)\cap B(x_j,k\epsilon))^2\times[0,1]} = \sigma^{j,k\epsilon}|_{(B(x_i,k\epsilon)\cap B(x_j,k\epsilon))^2\times[0,1]}$ for all $i, j \in I$. Then the nerve K of the cover $\boldsymbol{\mathcal{U}}$ is a locally finite simplicial complex of the same bounded homotopy type as X.*

Moreover, if the space X is proper, geodesic and admits a cocompact group action via isometries, and $\dim K < \infty$, then the constructed bounded homotopy equivalences $f\colon X \to K$ and $g\colon K \to X$ are quasi-isometries.

Remark 8.4. If the space X admits a conical bicombing σ, then one may construct the families $\{\sigma^{i,\epsilon} : i \in I\}$ and $\{\sigma^{i,3\epsilon} : i \in I\}$ satisfying the properties required for them in the statement above by restricting σ to appropriate balls.

Proof. The proof that X and the nerve of $\boldsymbol{\mathcal{U}}$ are of the same bounded homotopy type can be done using the construction from the proof of [BH99, Lemma I.7A.15], with the change that instead of using the unique geodesic between a pair of points $x, x' \in B(x_i, k\epsilon)$, where $i \in I$ and $k \in \{1, 3\}$, one may use the segment $t \mapsto \sigma^{i,k\epsilon}(x, x', t)$. We present its outline below.

Denote by v_i the vertex in K corresponding to the ball $B(x_i, \epsilon) \in \boldsymbol{\mathcal{U}}$.

The map $f\colon X \to K$ is constructed via a partition of unity, which is almost subordinate — it is subordinate upon ignoring the (topological) boundaries of the supports — to the (locally finite) open cover $\boldsymbol{\mathcal{U}}$: given $x \in X$ and $i \in I$, define $\varphi_i(x) := \max(0, \epsilon - d(x_i, x))$, and define $f(x)$ to have the v_i-coordinate equal to $\varphi_i(x) / \sum_{j\in I} \varphi_j(x)$. Observe that for any $i \in I$ we have the inclusion $f(B(x_i, \epsilon)) \subseteq \mathrm{st}(v_i)$, where $\mathrm{st}(v) = \bigcup\{\operatorname{int}\Delta : v \in \Delta,\ \Delta \text{ is a simplex of } K\}$ is the open star of the vertex v in K. In particular, $f(B(x_i, \epsilon)) \subseteq B(f(x_i), 2)$.

The map $g\colon K \to X$ is constructed inductively over the skeleta of K, maintaining the property that for each v_i we have that $g(K^{(d)} \cap \mathrm{St}_{\max}(v_i)) \subseteq B(x_i, \epsilon)$, where $K^{(d)}$ is the d-skeleton of K and $\mathrm{St}_{\max}(v_i)$ consists of $y \in K$ whose v_i-coordinate is not smaller than any other of its v_j-coordinates (where j ranges over I). Put $g(v_i) := x_i$. Assume that we have defined g on $K^{(d)}$. We shall extend it to $K^{(d+1)}$ for each $(d+1)$-simplex Δ in K separately. Let y_c be the central point of Δ and pick any point $c \in \bigcap\{B(x_j, \epsilon) : v_j \in \Delta^{(0)}\}$. Given a point $y \in \Delta^{(d)} \cap \mathrm{St}_{\max}(v_j)$, where $v_j \in \Delta^{(0)}$, one may define g to map the segment $[y, y_c]$ via the map $ty + (1-t)y_c \mapsto \sigma^{j,\epsilon}(g(y), c, t) \in B(x_j, \epsilon)$, where $t \in [0, 1]$. This gives a well-defined continuous extension of g to the whole Δ, as the functions $\sigma^{j,\epsilon}$ are assumed to agree with each other on intersections of their domains, are continuous and their domains are open in $X \times X \times [0, 1]$, and the considered segments $[y, y_c]$ cover the whole Δ. Observe that for any $i \in I$ we have that

$$
\begin{aligned}
g(\mathrm{st}(v_i)) &\subseteq g\big(\bigcup\{\mathrm{St}_{\max}(v_j) : v_j = v_i \text{ or } \{v_i, v_j\} \in K^{(1)}\}\big) \\
&\subseteq g\big(\bigcup\{\mathrm{St}_{\max}(v_j) : d(x_i, x_j) < 2\epsilon\}\big) \subseteq \bigcup\{B(x_j, \epsilon) : d(x_i, x_j) < 2\epsilon\} \subseteq B(x_i, 3\epsilon).
\end{aligned}
$$

Regarding the bounded homotopy between $g \circ f$ and the identity of X, observe that for any $i \in I$ we have that $g(f(B(x_i,\epsilon))) \subseteq g(\mathrm{st}(v_i)) \subseteq B(x_i, 3\epsilon)$. Therefore, for a point $x \in B(x_i,\epsilon)$, we may define the desired bounded homotopy to contain the map $(x,t) \mapsto \sigma^{i,3\epsilon}(g(f(x)),x,t)$. Similarly as above, it is a well-defined bounded homotopy due to the assumptions on the functions $\sigma^{i,3\epsilon}$.

Regarding the bounded homotopy between $f \circ g$ and the identity of K, observe that for any simplex Δ of K we have that

$$f(g(\Delta)) \subseteq f\big(g\big(\bigcup\{\mathrm{St}_{\max}(v_j) : v_j \in \Delta^{(0)}\}\big)\big) \subseteq f\big(\bigcup\{B(x_j,\epsilon) : v_j \in \Delta^{(0)}\}\big) \subseteq \bigcup\{\mathrm{st}(v_j) : v_j \in \Delta^{(0)}\}.$$

It is now a standard fact that one may construct a homotopy $\mathfrak{h}^K$ between $f \circ g$ and id_K such that for each simplex Δ of K the homotopy $\mathfrak{h}^K$ moves points of Δ along piecewise-linear segments contained in the union of open stars of vertices of Δ; in particular, $\mathfrak{h}^K$ is a bounded homotopy.

Regarding the proof of the 'moreover' part, note that we have proved above that the compositions $g \circ f$ and $f \circ g$ are at finite distance from the identity maps on X and K, respectively, therefore it is sufficient to prove that f and g are coarsely Lipschitz.

Regarding the proof for f, first note that we have that for any compact $A \subseteq X$ there exists a constant C_A such that the cardinality of any ϵ-net in A is bounded by C_A. Indeed, for any $(\epsilon/2)$-net N and ϵ-net M in A we have that each ϵ-ball centred in an element of M contains an element of N, and each element of N is contained in at most one ϵ-ball centred in an element of M; therefore $|N| \geq |M|$, so it is sufficient to take C_A equal to the cardinality of any $(\epsilon/2)$-net in A.

Our first goal is to show that $\sup\{d(f(x), f(x')) : x, x' \in X, d(x,x') \leq 1\} < \infty$. Take any $o \in X$. Since X admits a cocompact group action, there exists $R > 0$ such that the translates of $B(o,R)$ cover X. Let $C := C_{\overline{B}(o,R+1+\epsilon)}$ be as in the paragraph above. Let $x, x' \in X$ be such that $d(x,x') \leq 1$. By picking a maximal subset of points of pairwise distances not smaller than ϵ from the set $\{x_i : i \in I, B(x_i,\epsilon) \cap \overline{B}(x,1) \neq \emptyset\}$, one obtains a set $I_\odot \subseteq I$ such that $|I_\odot| \leq C$, as any group element that translates x into $B(o,R)$ translates the set $\{x_i : i \in I, B(x_i,\epsilon) \cap \overline{B}(x,1) \neq \emptyset\}$ into $B(o, R+1+\epsilon)$. Since one may connect x and x' with a geodesic contained in $\overline{B}(x,1)$, there exists a chain of points $x_{i_1}, \ldots, x_{i_k} \in B(x, 1+\epsilon)$ such that $i_j \in I$ for $1 \leq j \leq k$, $B(x_{i_j},\epsilon) \cap B(x_{i_{j+1}},\epsilon) \neq \emptyset$ for $1 \leq j < k$, $x \in B(x_{i_1},\epsilon)$ and $x' \in B(x_{i_k},\epsilon)$. Observe that for each i_j, where $1 \leq j \leq k$, there exists $i_j^\odot \in I_\odot$ such that $d(x_{i_j}, x_{i_j^\odot}) < \epsilon$; in particular, $f(x_{i_j^\odot})$ belongs to $B(f(x_{i_j}),2)$, thus $f(B(x_{i_j},\epsilon)) \subseteq B(f(x_{i_j}),2) \subseteq B(f(x_{i_j^\odot}),4)$. Then the chain of points $x_{i_1^\odot}, \ldots, x_{i_k^\odot}$ is such that $i_j^\odot \in I_\odot$ for $1 \leq j \leq k$, $B(f(x_{i_j^\odot}),4) \cap B(f(x_{i_{j+1}^\odot}),4) \neq \emptyset$ for $1 \leq j < k$, $f(x) \in B(f(x_{i_1^\odot}),4)$ and $f(x') \in B(f(x_{i_k^\odot}),4)$. By taking the shortest among such chains, one may assume that $k \leq |I_\odot| \leq C$; then we have that $d(f(x), f(x')) \leq 4k \leq 4C$. Finally, for arbitrary $x, x' \in X$, by considering a geodesic between x and x', and the triangle inequality, one may obtain that $d(f(x), f(x')) \leq 4C\lceil d(x,x') \rceil \leq 4Cd(x,x') + 4C$, where $\lceil \cdot \rceil$ is the ceiling function.

Regarding the proof for g, recall the identification of d-simplices with appropriate subset of $\mathbb{R}^{d+1}$ with the supremum metric, and observe that for any $i \in I$, assigning to an element of the open star $\mathrm{st}(v_i)$ the value of its v_i-coordinate is a 1-Lipschitz function. Therefore, for any $y \in K$ and $i \in I$ such that $y \in \mathrm{st}(v_i)$ we have that $B(y, \lambda_i) \subseteq \mathrm{st}(v_i)$, where λ_i is the value of the v_i-coordinate of y. Therefore, as $\dim K < \infty$, for each $y \in K$ there exists a vertex v_{i_y} such that $B(y, 1/\dim(K)) \subseteq \mathrm{st}(v_{i_y})$. Consider $y, y' \in K$. If $d(y,y') \leq 1/(2\dim(K))$, then $d(f(y), f(y')) \leq 6\epsilon$ (as $f(\mathrm{st}(v_{i_y})) \subseteq B(x_{i_y}, 3\epsilon)$). Therefore for arbitrary $y, y' \in K$ we have that

$$d(f(y), f(y')) \leq \left\lceil \frac{d(y,y')}{(2\dim(K))^{-1}} \right\rceil \cdot 6\epsilon \leq 12\dim(K)\epsilon d(y,y') + 6\epsilon. \qquad \square$$

The default cohomology theory in this section is the Alexander–Spanier cohomology $\bar{H}^*$. We note that all of the reasonings in the remaining part of this subsection also work with the simplicial cohomology in the place of the Alexander–Spanier cohomology; the extra properties of the latter cohomology theory, mainly the consequences of admitting more so-called taut pairs, see [Spa94, above 6.1.7, and Section 6.6], will be used mainly in Remark 8.7 and the proof of Theorem 8.8.

For a topological space X, the *(Alexander–Spanier) cohomology with compact support* $\bar{H}^*_c(X)$ is defined as the direct limit of the system $\{\bar{H}^*(X, X \setminus K) : K \subseteq X \text{ compact}\}$ (with the homomorphism in this system being the maps induced by inclusions), see [Spa94, Theorem 6.6.15]. We note here that excision, [Spa94, Theorem 6.4.4], allows us to view the groups $\bar{H}^*_c(V)$, where $V \subseteq X$ is open, also as the following direct limit:

$$\bar{H}^*_c(V) \cong \varinjlim \{\bar{H}^*(X, X \setminus K) : K \subseteq V \text{ compact}\}. \tag{8-1}$$

Let X be a proper metric space, $i \in \mathbb{N}$ and T be a function $[0, +\infty) \to [0, +\infty)$. Then we say that the group $\bar{H}^i_c(X)$ of cohomology with compact support is:

- *T-uniformly trivial*, if the map $H^i_c(B(x,r)) \to H^i_c(B(x, r + T(r)))$ induced by (the system of) inclusions, recall (8-1), is trivial;
- *uniformly trivial*, if it is T-uniformly trivial for some function T;
- *T-neighbourhood-uniformly trivial*, if for each $r > 0$ and compact set A contained in a closed ball of radius r the map $\bar{H}^i(X, X \setminus A) \to \bar{H}^i(X, X \setminus \overline{B}(A, T(r)))$ induced by inclusion is trivial.

Let K be a locally finite simplicial complex. By the *simplicial cohomology with compact support* $H^*_{\mathrm{spl},c}(K)$ we mean the one resulting from the (co)chain complex

$$C^*_{\mathrm{spl},c}(K) = \{\varphi \in C^*_{\mathrm{spl}}(K) : \varphi \text{ is supported in a finite subcomplex of } K\}.$$

For $i \in \mathbb{N}$ and a function $T\colon [0,\infty) \to [0,\infty)$, we say that the group $H^i_{\mathrm{spl},c}(K)$ is *T-neighbourhood-uniformly trivial* if for each $r > 0$ and i-cocycle φ supported in a finite subcomplex L contained in a closed ball of of radius r there exists an $(i-1)$-cochain ψ supported in (a, necessarily finite, subcomplex of K contained in) the compact set $\overline{B}(L, T(r))$. (Recall that we equip simplicial complexes with a piecewise–unit-ℓ^∞ metric; however, in the remainder of this section, we only really use the fact that the metric on a simplicial complex is such that each simplex is of diameter at most 1, and that Lemma 8.3 holds.)

Lemma 8.5. *(i) Let X be a proper metric space. If $\bar{H}^i_c(X)$ is T-uniformly trivial, then it is $(r \mapsto 2T(r+1) + 2r + 2)$-neighbourhood-uniformly trivial.*

(ii) Let (X, d_X) and (Y, d_Y) be proper metric spaces. Let $f\colon X \to Y$ and $g\colon Y \to X$ be bounded homotopy equivalences. Assume that f and g are quasi-isometries, and that $\bar{H}^i_c(Y)$ is T-neighbourhood-uniformly trivial. Then $\bar{H}^i_c(X)$ is $(r \mapsto CT(Cr + 2C^2) + 2C)$-neighbourhood-uniformly trivial for some $C > 0$.

(iii) Let K be a locally finite simplicial complex and $T\colon [0,+\infty) \to [0,+\infty)$. (a) If $\bar{H}^i_c(X)$ is T-neighbourhood-uniformly trivial, then $H^i_{\mathrm{spl},c}(X)$ is $(r \mapsto T(r+1)+3)$-neighbourhood-uniformly trivial. (b) If $H^i_{\mathrm{spl},c}(X)$ is T-neighbourhood-uniformly trivial, then $\bar{H}^i_c(X)$ is $(r \mapsto T(r+1)+4)$-neighbourhood-uniformly trivial.

Proof. (i) Consider $\emptyset \neq A \subseteq \overline{B}(x, r)$. Then the map

$$\bar{H}^i(X, X \setminus A) \to \bar{H}^i(X, X \setminus \overline{B}(x, T(r+1) + r + 1)),$$

in view of (8-1), factors as

$$\begin{aligned}\bar{H}^i(X, X \setminus A) \to \bar{H}_c(B(x, r+1)) &\to \bar{H}_c(B(x, T(r+1) + r + 1)) \\ &\to \bar{H}^i(X, X \setminus \overline{B}(x, T(r+1) + r + 1)),\end{aligned}$$

therefore is trivial, as the middle arrow is trivial. The claim follows, as $\overline{B}(x, T(r+1)+r+1)$ is of diameter at most $2(T(r+1)+r+1)$, hence

$$A \subseteq \overline{B}(x, T(r+1) + r + 1) \subseteq \overline{B}(A, 2(T(r+1) + r + 1)).$$

(ii) Denote by $\mathfrak{h}^X\colon X \times [0,1] \to X$ the bounded homotopy between $g \circ f$ and id_X. Combining various assumptions of this lemma, we obtain that there exists a constant $C > 0$ such that the diameter of the set $\mathfrak{h}^X(\{x\} \times [0,1])$ is smaller than C for all $x \in X$

(in particular, the maps $g \circ f$ and id_X are C-close), the image of g is C-dense in Y, and for all $y, y' \in Y$ the inequality $C^{-1}d_Y(y,y') - C \le d_X(g(y), g(y')) \le C d_Y(y,y') + C$ holds.

For any compact set $A \subseteq X$ contained in a ball $B(x_A, r)$ we have the following diagram.

$$\begin{tikzcd}
\bar{H}^i(X, X\setminus A) \arrow[r, "g^*"] \arrow[dd, "\mathrm{id}^*=(gf)^*"'] \arrow[drr, "(gf)^*"] & \bar{H}^i(Y, Y\setminus g^{-1}(A)) \arrow[r] & \bar{H}^i(Y, Y\setminus \overline{B}(g^{-1}(A), T(Cr+2C^2))) \arrow[d, "f^*"] \\
 & & \bar{H}^i(X, X\setminus f^{-1}(\overline{B}(g^{-1}(A), T(Cr+2C^2)))) \arrow[d] \\
\bar{H}^i(X, X\setminus \overline{B}(A, C)) \arrow[rr] & & \bar{H}^i(X, X\setminus \overline{B}(A, CT(Cr+2C^2)+2C))
\end{tikzcd}$$

All unlabelled arrows are induced by inclusions. The maps $(gf)^*, \mathrm{id}^*\colon \bar{H}^i(X, X\setminus A) \to \bar{H}^i(X, X\setminus \overline{B}(A, C))$ are equal by the fact that the bounded homotopy $\mathfrak{h}^X$ between $g \circ f$ and id_X induces a homotopy between the maps $(X, X\setminus \overline{B}(A,C)) \to (X, X\setminus A)$ induced by $g \circ f$ and id_X (see [Spa94, Theorem 6.5.6]). The vertical map induced by inclusion in the second column is well-defined by the following argument. Denote by A' the set $f^{-1}(\overline{B}(g^{-1}(A), T(Cr+2C^2)))$, and consider $x' \in A'$. Then there exists $y \in Y$ such that $d_Y(y, f(x')) \le T(Cr+2C^2)$ and $g(y) \in A$. Then we have that

$$\begin{aligned}
d_X(x', A) \le d_X(x', g(y)) &\le d_X(x', g(f(x'))) + d_X(g(f(x')), g(y)) \\
&\le C + C d_Y(f(x'), y) + C \le CT(Cr+2C^2) + 2C,
\end{aligned}$$

so the discussed map in the diagram is indeed well-defined.

The map in the first row that is induced by inclusion is zero, as the set $g^{-1}(A)$ is contained in a ball of radius $Cr + 2C^2$: let $y_A \in Y$ be such that $d_X(g(y_A), x_A) \le C$; then

$$g^{-1}(A) \subseteq g^{-1}(B(x_A, r)) \subseteq g^{-1}(B(g(y_A), r + C)) \subseteq B(y_A, C(r+C) + C^2).$$

Finally, joining various pieces together, we obtain that the map

$$\bar{H}^i(X, X\setminus A) \to \bar{H}^i(X, X\setminus \overline{B}(A, CT(Cr+2C^2)+2C))$$

induced by inclusion (the result of going the down-right route in the diagram) is trivial (as the result of going the right-right-down-down route in the diagram).

(iii) There is a canonical isomorphism between the simplicial cohomology and the Alexander–Spanier cohomology of a simplicial complex relative to its subcomplex, see e.g. [Spa94, Section 6.5 and Theorem 4.6.8]. For a subcomplex L of K we denote by L_{c} the subcomplex of K consisting of simplices contained in $K \setminus L$; observe that we have the following inclusions: $L_{\mathsf{c}} \subseteq K \setminus L$ and $K \setminus \overline{B}(L, 1) \subseteq L_{\mathsf{c}}$.

(a) Take $\varphi \in Z^i_{\mathrm{spl},c}(K)$ supported in a finite subcomplex L of K that is contained in a closed ball of radius r. Let L° be the finite subcomplex of K consisting of the simplices intersecting the set $\overline{B}(L, T(r+1)+1))$ (in particular, $L^{\circ} \subseteq \overline{B}(L, T(r+1)+2))$. We have the following diagram.

$$\begin{array}{ccccccc}
H^i_{\mathrm{spl}}(K, L_{\mathsf{c}}) & \longrightarrow & & & & & H^i_{\mathrm{spl}}(K, L^{\circ}_{\mathsf{c}}) \\
\downarrow\cong & & & & & & \downarrow\cong \\
\bar{H}^i(K, L_{\mathsf{c}}) & \longrightarrow & \bar{H}^i(K, K \setminus \overline{B}(L,1)) & \longrightarrow & \bar{H}^i(K, K \setminus \overline{B}(L, T(r+1)+1)) & \longrightarrow & \bar{H}^i(K, L^{\circ}_{\mathsf{c}})
\end{array}$$

The horizontal arrows are induced by inclusions. The vertical arrows are canonical isomorphisms between cohomology theories. By the assumption, the middle arrow in the second row is zero, therefore the arrow in the first row is zero. One has that $K \setminus L^{\circ}_{\mathsf{c}} \subseteq \overline{B}(L^{\circ}, 1) \subseteq \overline{B}(L, T(r+1)+3)$, which finishes the proof.

(b) Let $A \subseteq K$ be compact and contained in a closed ball of radius r. Let L be the smallest subcomplex of K that contains A. Observe that $L \subseteq \overline{B}(A, 1)$, therefore is contained in a ball of radius $r+1$. Let L° be the smallest subcomplex of K containing $\overline{B}(L, T(r+1)+1)$ (so, in particular, $L^{\circ} \subseteq \overline{B}(L, T(r+1)+2) \subseteq \overline{B}(A, T(r+1)+3)$). We have the following diagram.

$$\begin{array}{ccccccc}
\bar{H}^i(K, K \setminus A) & \longrightarrow & \bar{H}^i(K, L_{\mathsf{c}}) & \longrightarrow & \bar{H}^i(K, L^{\circ}_{\mathsf{c}}) & \longrightarrow & \bar{H}^i(K, K \setminus \overline{B}(A, T(r+1)+4)) \\
 & & \downarrow\cong & & \downarrow\cong & & \\
 & & H^i_{\mathrm{spl}}(K, L_{\mathsf{c}}) & \longrightarrow & H^i_{\mathrm{spl}}(K, L^{\circ}_{\mathsf{c}}) & &
\end{array}$$

The horizontal arrows are induced by inclusions. The vertical arrows are canonical isomorphisms between cohomology theories. By the assumption, the arrow in the second row is zero, therefore composition of the arrows in the top row is also zero. □

Lemma 8.6 (cf. [GO07, Theorem 2, Proposition 1 and its Corollary])**.** *Let X be a finite-dimensional proper metric space that admits a* ccc *geodesic bicombing σ and a cocompact group action via isometries.*

(i) There is a finite-dimensional, countable, locally finite simplicial complex K such that X and K are of the same bounded homotopy type. Moreover, the bounded homotopy equivalences justifying this fact may be chosen so that they are quasi-isometries.

(ii) If $\bar{H}^i_c(X)$ is trivial, then it is uniformly trivial.

(iii) Assume that $\bar{H}^i_c(X)$ is trivial for all $i > k$. Then there exists a number $\bar{t}$ such that the cohomology groups $\bar{H}^i_c(X)$ for $i > k$ are $(r \mapsto \bar{t})$-neighbourhood-uniformly trivial, i.e. for all compact $A \subseteq X$ and $i > k$ the map $\bar{H}^i(X, X \setminus A) \to \bar{H}^i(X, X \setminus \overline{B}(A, \bar{t}))$ induced by inclusion is trivial.

(iv) Under the assumptions of (iii), there exists a number t such that the maps $\bar{H}^i_c(U) \to \bar{H}^i_c(B(U,t))$ (induced by the system of inclusions) are trivial for all open subsets $U \subseteq X$ and $i > k$.

Proof. (i) Consider a maximal set $E \subseteq X$ such that for each pair of different points $x, x' \in E$ we have $d(x, x') \geq 1$. Then the family $\{B(x,1) : x \in E\}$ is an open cover of X. Denote by K its nerve. Then, by the first two paragraphs of the proof of [GO07, Theorem 2], we have that K is countable, locally finite and finite-dimensional. The remaining properties of the complex K follow by Lemma 8.3 (see Remark 8.4).

(ii) The proof of [GO07, Proposition 1] may be applied directly, since the CAT(0)-assumption is used only to deduce the conclusion of (i).

(iii) We are working with the following picture: $\bar{H}^*_c(X) \cong \bar{H}^*_c(K) \cong H^*_{\mathrm{spl},c}(K)$, where K is the simplicial complex from (i). Put $n := \dim K$. By (ii), the bounded cohomology groups $\bar{H}^i_c(X)$ for $i > k$ are uniformly trivial, and by a subsequent application of Lemma 8.5(i), (ii) and (iii)(a), the bounded cohomology groups $H^i_{\mathrm{spl},c}(K)$ for $i > k$ are T_i-neighbourhood-uniformly trivial for some functions T_i, so we may pick a function T such that each simplicial i-cocycle for $k < i \leq n$ with support contained in a closed ball of radius r cobounds in the closed $T(r)$-neighbourhood of its support. Now it suffices to show that there exists a number $\bar{t}$ such that for all $i > k$ each simplicial i-cocycle in K of bounded support cobounds in the closed $\bar{t}$-neighbourhood of its support, as then the claim follows from subsequent application of Lemma 8.5(iii)(b) and (ii).

To this end, we construct a chain homotopy $D = D^i \colon C^i_{\mathrm{spl},c}(K) \to C^{i-1}_{\mathrm{spl},c}(K)$ between the identity map id and the zero map for $i > k$ that satisfies an additional property: for each i there exists a number $\tau(i)$ such that for each $c \in C^i_{\mathrm{spl},c}(K)$ we have that $D^i c$ is supported in the closed $\tau(i)$-neighbourhood of c. First, define $D^i = 0$ and take $\tau(i) = 0$ for $i > n$. This clearly satisfies the required properties. Next, take i such that $k < i \leq n$, assume that we have defined D^{i+1} and $\tau(i+1)$, and let c be an i-cochain that is supported in a simplex. Then D^i must satisfy $c = \delta D^{i+1} c + D^i \delta c$, therefore $\delta D^i c = c - D^{i+1}\delta c$. Observe that the right-hand side is a cocycle:

$$\delta(c - D^{i+1}\delta c) = \delta c - (\delta D^{i+1})(\delta c) = \delta c - \delta c + D^{i+2}(\delta\delta c) = 0 + D^{i+2}(0) = 0,$$

therefore it cobounds in the closed $T(\tau(i+1)+2)$-neighbourhood of its support, since, by the assumption, it is supported in the closed $(\tau(i+1)+1)$-neighbourhood of c. Define $D^i c$ to be any $(i-1)$-cochain that is supported in the closed $T(\tau(i+1)+2)$-neighbourhood of the support of c and satisfies the equation $\delta D^i c = c - D^{i+1}\delta c$. Next, extend D^i linearly to $C^i_{\mathrm{spl},c}(K)$. One may easily verify that it suffices to take $\tau(i) := T(\tau(i+1)+2)$.

Finally, let $\bar{t} := \max_{i>k} \tau(i)$. Given a simplicial i-cocycle c, we have that $c = \delta Dc + D\delta c = \delta Dc$, and, by the construction, Dc is supported in the closed $\bar{t}$-neighbourhood of c, as required.

(iv) Statement (iii) gives triviality of the maps $\bar{H}^i(X, X \setminus A) \to \bar{H}^i(X, X \setminus \overline{B}(A, \bar{t}\,))$

for $A \subseteq U$ compact, which, in view of (8-1), implies the triviality of the limit map $\bar{H}^i_c(U) \to \bar{H}^i_c(B(U, \bar{t}+1))$. Therefore it is sufficient to take $t := \bar{t}+1$. $\square$

8.2. Main lemmas and proof of Theorem 8.1 (Thm. X)

Remark 8.7 (cf. [GO07, Remark 2]). Let X be a proper metric space that admits a $\mathfrak{ccc}$ geodesic bicombing σ. Then $\bar{H}^{*+1}_c(X)$ and the reduced Alexander–Spanier cohomology group $\tilde{\bar{H}}^*(\partial_\sigma X)$ are isomorphic. Indeed, fix $o \in X$; then for any $R > 0$ and $i \in \mathbb{N}$ we have the following fragment of the exact sequence of the pair $(\overline{X}_\sigma, \overline{X}_\sigma \setminus \overline{B}_X(o,R))$:

$$\bar{H}^i(\overline{X}_\sigma) \to \bar{H}^i(\overline{X}_\sigma \setminus \overline{B}_X(o,R)) \to \bar{H}^{i+1}(\overline{X}_\sigma, \overline{X}_\sigma \setminus \overline{B}_X(o,R)) \to \bar{H}^{i+1}(\overline{X}_\sigma).$$

For $i \geq 1$ the first and the last of these groups are trivial, as $\overline{X}_\sigma$ is contractible; therefore the middle arrow is an isomorphism, which together with excision gives that

$$\bar{H}^i(\overline{X}_\sigma \setminus \overline{B}_X(o,R)) \cong \bar{H}^{i+1}(\overline{X}_\sigma, \overline{X}_\sigma \setminus \overline{B}_X(o,R)) \cong \bar{H}^{i+1}(X, X \setminus \overline{B}_X(o,R));$$

passing to the limit, we obtain the desired isomorphism (see [Spa94, Theorem 6.6.2]). Similarly follows the case of $i = 0$, where we have $\mathbb{Z}$ as the left term of the exact sequence above.

In fact, the above argument works in a more general setting. Consider a compact subset Z of a compact space $\mathfrak{X}$, such that Z is a Z-set in $\mathfrak{X}$. Let $\{\mathfrak{h}_t \colon \mathfrak{X} \to \mathfrak{X} : t \in [0,1]\}$ be the homotopy from the definition of Z-set. Assume that $\mathfrak{X}$ is contractible and $\bigcup_{t>0} \mathfrak{h}_t(\mathfrak{X}) = \mathfrak{X} \setminus Z$. Then $\bar{H}^{*+1}_c(\mathfrak{X} \setminus Z) \cong \tilde{\bar{H}}^*(Z)$ — it suffices to consider the sets $(\mathfrak{h}_t(\mathfrak{X}))_{t>0}$ in the place of balls $(\overline{B}(o,R))_{R>0}$ in the argument above, as the family $\{\mathfrak{h}_t(\mathfrak{X}) : t > 0\}$ is cofinal in the family of all compact subsets of $\mathfrak{X} \setminus Z$.

Theorem 8.8 (Theorem IX; cf. [GO07, Main Theorem]). *Let X be a non-compact finite-dimensional proper metric space that admits a $\mathfrak{ccc}$ geodesic bicombing σ and a cocompact group action via isometries. Then the reduced Alexander–Spanier cohomology group $\tilde{\bar{H}}^{\dim \partial_\sigma X}(\partial_\sigma X)$ is non-zero.*

Remark 8.9. Recall that in the proof of Theorem I in Section 2 we proved that for a finite-dimensional proper metric space that admits a $\mathfrak{ccc}$ bicombing we have that $\dim \overline{X}_\sigma \leq \dim X$. In particular, $\dim \partial_\sigma X < \infty$, so the cohomology group $\tilde{\bar{H}}^{\dim \partial_\sigma X}(\partial_\sigma X)$ in the statement above is well-defined.

Remark 8.10 (cf. [GO07, below Remark 3]). If the action of the group, G, in the statement above is geometric (i.e. it is additionally proper), then we have the isomorphism $\bar{H}^*_c(X) \cong H^*(G, \mathbb{Z}G)$, which by Remark 8.7 gives the isomorphism $\tilde{\bar{H}}^*(\partial_\sigma X) \cong H^{*+1}(G, \mathbb{Z}G)$. Indeed, by taking the nerve of a locally finite open cover $\mathcal{U}$ of X by balls (of the same radius), such that $\mathcal{U}$ is closed under the action of G, one obtains a simplicial complex K with a geometric (and simplicial) action of G, boundedly homotopic to the space X (see Lemma 8.3). Then by [Bro82, Exercise VIII.7.4] it follows that $H^*(G, \mathbb{Z}G) \cong \bar{H}^*_c(K) \cong \bar{H}^*_c(X)$.

Proof. There is the following cohomological definition of dimension of a topological space Y:

$$\dim_{\mathbb{Z}} Y = \sup\{i \in \mathbb{N} : (\exists A \subseteq Y)(A \text{ is closed}, \bar{H}^i(Y, A) \neq 0)\}.$$

(Usually one uses the Čech cohomology in the above definition, but these coincide with the Alexander–Spanier cohomology — see [Spa94, Corollary 6.8.8 and Exercise 6.D] or [Dow52, Theorem 2].) The studies on (co)homological notions of dimension date back to Alexandrov [Ale32]. The above definition of dimension coincides with the covering dimension for separable metric spaces of finite covering dimension, see e.g. [Eng78, below Corollary 1.9.9].

Since the space $\overline{X}_\sigma$ is of finite dimension (see Remark 8.9), the topological dimension n of $\partial_\sigma X$ is also finite, therefore its cohomological dimension $\dim_{\mathbb{Z}} \partial_\sigma X$ is also n. Let d be the largest number such that $\bar{H}^{d+1}_c(X) \cong \tilde{\bar{H}}^d(\partial_\sigma X) \neq 0$ (recall Remark 8.7 for the isomorphism). By considering the set A consisting of a single point (in the definition of the cohomological dimension above), we have $d \leq n$. Assume a contrario that $d < n$. Take $\emptyset \neq A \subseteq \partial_\sigma X$ closed such that $\bar{H}^n(\partial_\sigma X, A) \neq 0$ (for the non-emptiness of A, one may use the exact sequence of the pair (X, A) for $n \geq 1$, or [Spa94, Theorem 6.4.5] for $n = 0$). Fix a basepoint $o \in X$. By [GO07, Lemma 1], there exist open balls $B_k \subseteq \partial_\sigma X \setminus A$ of radius $1/k$ (with respect to some metric on the boundary $\partial_\sigma X$, it does not matter which one) for sufficiently large $k \in \mathbb{N}$, and $\bar{\gamma} \in \partial_\sigma X \setminus A$, such that the balls B_k converge to $\bar{\gamma}$, and the maps $\bar{H}^n(\partial_\sigma X, \partial_\sigma X \setminus B_k) \to \bar{H}^n(\partial_\sigma X, A)$ induced by inclusion are all non-zero.

The outline of the remaining part of the proof of the current theorem is as follows. Using the Cone_o operation, one may deduce from the non-zeroness of the map $\bar{H}^n(\partial_\sigma X, \partial_\sigma X \setminus B_k) \to \bar{H}^n(\partial_\sigma X, A)$ for a sufficiently large k — where we have the set A contained in a 'very large' set $\partial_\sigma X \setminus B_k$ — that we have a similar situation in X, namely that the map $\bar{H}^{n+1}_c(X \setminus E) \to \bar{H}^{n+1}_c(X \setminus D)$ induced by inclusion is non-zero for some $E, D \subseteq X$ such that a ball of large diameter around $X \setminus E$ is contained in $X \setminus D$. This, however, contradicts Lemma 8.6(iv).

Now we proceed to the details. Let the set D be related to the set A as in the statement of Lemma 8.2. Pick a constant t as guaranteed by Lemma 8.6(iv). Since $\bar{\gamma} \notin A$, by compactness of A and Proposition 2.1(ii), one may choose $s_{\bar{\gamma}}$ such that $d(\varrho_{o,\bar{\gamma}}(s_{\bar{\gamma}}), \mathrm{Cone}_o(A)) \geq t + 3$. Since B_k converges to $\bar{\gamma}$, one may choose k such that

$\exp_o(B_k \times \{s_{\bar\gamma}\}) \subseteq B(\varrho_{o,\bar\gamma}(s_{\bar\gamma}), 1)$. Let $B := B_k$. Define

$$E := \overline{B}(o, s_{\bar\gamma}) \cup \{\bar x \in \overline{X}_\sigma : d(\varrho_{o,\bar x}(s_{\bar\gamma}), \varrho_{o,\bar\gamma}(s_{\bar\gamma})) \geq 1\} \cup \mathrm{Cone}_o(\partial_\sigma X \setminus B).$$

Observe that it is a closed set such that $E \cap \partial_\sigma X = \partial_\sigma X \setminus B$. Furthermore, $d(D \cap X, X \setminus E) \geq t+1$: take $a \in \mathrm{Cone}_o(A) \cap X$ and $x \in X \setminus E$; then $d(x,o) > s_{\bar\gamma}$, so by conicality of σ we have that

$$\begin{aligned} d(a,x) &\geq d(\sigma_{o,a}(s_{\bar\gamma}/d(o,x)), \sigma_{o,x}(s_{\bar\gamma}/d(o,x))) = d(\sigma_{o,a}(s_{\bar\gamma}/d(o,x)), \varrho_{o,x}(s_{\bar\gamma})) \\ &\geq d(\sigma_{o,a}(s_{\bar\gamma}/d(o,x)), \varrho_{o,\bar\gamma}(s_{\bar\gamma})) - d(\varrho_{o,\bar\gamma}(s_{\bar\gamma}), \varrho_{o,x}(s_{\bar\gamma})) \geq t+3-1 = t+2; \end{aligned}$$

the claim follows as $D \cap X \subseteq \overline{B}(\mathrm{Cone}_o(A) \cap X, 1)$.

We have the following diagram.

$$\begin{tikzcd}
 & \bar H^n(\partial_\sigma X \cup E, E) \arrow[r] \arrow[dddd] & \bar H^{n+1}(\overline{X}_\sigma, \partial_\sigma X \cup E) \arrow[dddd] & & \\
\bar H^n(\partial_\sigma X, \partial_\sigma X \setminus B) \arrow[ur, "\cong"] \arrow[dd] & & & \bar H_c^{n+1}(X \setminus E) \arrow[ul, "\cong"'] \arrow[d] & \\
 & & & \bar H_c^{n+1}(B(X \setminus E, t+1)) \arrow[d] & \\
\bar H^n(\partial_\sigma X, A) \arrow[dr, "\cong"] & & & \bar H_c^{n+1}(X \setminus D) \arrow[dl, "\cong"'] & \\
0 = \bar H^n(\overline{X}_\sigma, D) \arrow[r] & \bar H^n(\partial_\sigma X \cup D, D) \arrow[r, "\cong"] & \bar H^{n+1}(\overline{X}_\sigma, \partial_\sigma X \cup D) \arrow[rr] & & \bar H^{n+1}(\overline{X}_\sigma, D) = 0
\end{tikzcd}$$

The upper and the lower rows are fragments of the exact sequences of the triples $(\overline{X}_\sigma, E \cup \partial_\sigma X, E)$ and $(\overline{X}_\sigma, D \cup \partial_\sigma X, D)$, respectively. In the lower row, the middle arrow is an isomorphism, since D is a deformation retract of $\overline{X}_\sigma$. The two long vertical arrows in the middle are induced by the inclusion of triples $(\overline{X}_\sigma, D \cup \partial_\sigma X, D) \hookrightarrow (\overline{X}_\sigma, E \cup \partial_\sigma X, E)$. The square on the left is the excision of X (the so-called strong excision property, see [Spa94, Theorem 6.6.5]). The isomorphisms on the right follow by the fact that, in compact spaces, the cohomology relative to a closed set may be identified with the compactly supported cohomology of its complement, see [Spa94, Lemma 6.6.11]. The vertical arrows on the right follow from the inclusions $X \setminus E \subseteq B(X \setminus E, t+1) \subseteq X \setminus D$.

We obtain a contradiction in the following way. The vertical arrow on the left is non-zero by the assumption on the sets B_k for $k \in \mathbb{N}$, therefore, moving step-by-step from the left to the right, one shows that each vertical arrow in the diagram is also non-zero. In particular, the map $\bar H_c^{n+1}(X \setminus E) \to \bar H_c^{n+1}(B(X \setminus E, t+1))$ is non-zero, which gives a contradiction with the definition of the number t. $\square$

We note that the argument in the proof of the lemma below works also for the simplicial cohomology.

Lemma 8.11 (cf. [Ont05, Theorem A]). *Let X be a proper non-compact metric space that admits a* ccc *geodesic bicombing σ and a cocompact group action via isometries. If any of the groups $\bar{H}^i_c(X)$ is non-zero, then X is almost σ-geodesically complete.*

Proof. The proof [Ont05, Theorem A] can be adapted by changing each occurrence of '(the unique) CAT(0)-geodesic (ray)' to '(the) σ-geodesic(/ray)'. In particular, the objects: ℓ_α (which later on we call L_α, to avoid confusing it with the function ℓ_o from Definition 2.3), $f^{p,s}\colon X \to X$, and $[p,x]$, $[p,x_0)$, where α is a σ-geodesic, $p,x \in X$, $s > 0$ and $x_0 \in \partial_\sigma X$, become

$$(L_\alpha =)\ell_\alpha = \sup\{t \in \mathbb{R} : \text{there exists a } \sigma\text{-geodesic of length } t \text{ that extends } \alpha\},$$

$f^{p,s}(x) = \exp_p(x, \max(d(p,x) - s, 0))$, $[p,x] = \varrho_{p,x}$ and $[p,x_0) = \varrho_{p,x_0}$. The last two notations will not be used further in this proof. The function $f^{p,s}$ is the key object in this proof, and the value $f^{p,s}(x)$ can be described as the point reached in the following walk: 'starting in x, go backwards along the σ-geodesic that begins in p and ends in x, towards p, with unit speed for time s, unless you reach p earlier — then stop'.

The outline is as follows. Assume that the space X is not almost σ-geodesically complete; we shall show that then every element $\varphi \in H^i(X, X \setminus \overline{B}(x,R))$, where $x \in X$ and $R > 0$, is zero in $\bar{H}^*_c(X)$ (i.e. the image of φ under the canonical map is zero). Fix a basepoint $o \in X$. The action of G on X via isometries induces an action of G on $\bar{H}^*_c(X)$ via isomorphisms. Therefore, since the action of G on X is cocompact, one may assume without loss of generality that the ball $\overline{B}(x,R)$ intersects no σ-ray originating in o: the G-orbit of x is D-dense in X for some $D > 0$, and, as X is not almost σ-geodesically complete, there exists $x_\odot \in X$ such that $\overline{B}(x_\odot, D+R)$ intersects no σ-ray; let g be such that $d(gx, x_\odot) \leq D$; then $g^*\varphi \in \bar{H}^i(X, X \setminus \overline{B}(gx,R))$ is non-zero in $\bar{H}^*_c(X)$ iff φ is, and $\overline{B}(gx,R)$ intersects no σ-ray. Using properness of X, one may obtain that $C := \sup\{L_{\varrho_{o,x'}} : x' \in \overline{B}(x,R)\}$ is finite: if there existed $x_n \in X$ such that $d(o,x_n) \to \infty$ and $t_n \leq d(o,x_n)$ such that $\varrho_{o,x_n}(t_n) \in \overline{B}(x,R)$, then by compactness of $\overline{B}(x,R)$ and $\overline{X}_\sigma$, one could choose a subsequence x_{n_k} convergent to some $\bar{x} \in \partial_\sigma X$ such that $\varrho_{o,x_{n_k}}(t_{n_k}) \to a$ for some $a \in \overline{B}(x,R)$; then $a = \lim_k \varrho_{o,x_{n_k}}(t_{n_k}) = \varrho_{o,\bar{x}}(d(o,a))$ (recall Proposition 2.4), which is a contradiction. Since the identity id_X and the function $f^{o,C}$ are boundedly homotopic, the (images under the canonical maps of the) elements $\varphi = (\mathrm{id}_X)^*\varphi \in \bar{H}^i(X, X \setminus \overline{B}(x,R))$ and $(f^{o,C})^*\varphi \in \bar{H}^i(X, X \setminus (f^{o,C})^{-1}(\overline{B}(x,R)))$ are equal in $\bar{H}^i_c(X)$, see [Spa94, Theorem 6.5.6]. Since the image of $f^{o,C}$ omits $\overline{B}(x,R)$, the latter cohomology group is trivial. Therefore φ is trivial in $\bar{H}^i_c(X)$. □

Proof. (of Theorem 8.1 (Thm. X)) Follows from Theorem 8.8 and Lemma 8.11, and an application of the isomorphism considered in Remark 8.7. □

9. Problems and open questions

Below we collect and present some problems and open questions arising from this article, which constitute a natural continuation of the topics of research discussed in this paper.

Q1. Does a counterpart of Theorem 4.1 (Thm. IV) hold for Helly groups? — that is: does there exist a group acting an two Helly graphs such that the boundaries of their injective hulls (recall the proof of Corollary II(ii)) are non-homeomorphic?

A discussion on adapting the example by Croke and Kleiner [CK00], used in the proof of Theorem 4.1, to the Helly case is made in Remark 4.9(ii), where it is pointed out that a problem lies in local non-Hellyness around diagonal gluing-lines — does a local Hellyfication around such lines solve the problem?

Q2. Does there exist a group acting geometrically on more than two (especially, uncountably many) injective metric spaces with pairwise non-homeomorphic boundaries?

A discussion on using the Wilson's approach [Wil05] to answer this question positively is made in Remark 4.9(i).

Q3. May the second conclusion of Corollary 5.4 (Thm. V) not hold for a locally finite CAT(0) cube complex of arbitrary dimension? — that is: does there exist a locally finite CAT(0) cube complex X such that the boundaries of X endowed with the (standard) CAT(0) piecewise-ℓ^2 metric and the injective piecewise-ℓ^∞ metric are non-homeomorphic?

May the first conclusion of Corollary 5.4 not hold for a locally finite CAT(0) cube complex (of arbitrary dimension) that admits a cocompact group action? — that is: does there exist a locally finite CAT(0) cube complex X admitting a cocompact group action such that the identity of X does not extend to a homeomorphism between the boundary-compactification of X endowed with the (standard) CAT(0) piecewise-ℓ^2 metric and the boundary-compactification of X endowed with the injective piecewise-ℓ^∞ metric?

A related discussion is made in Remark 5.10(i).

Q4. Do the results relating topological properties of boundaries of CAT(0) spaces and algebraic properties of groups acting upon them, e.g. in the spirit of Swenson [Swe99] or Papasoglu–Swensson [PS09], extend to the realm of spaces admitting $\mathfrak{ccc}$ geodesic bicombings? — for example: does the boundary of a $\mathfrak{ccc}$-bicombable space acted upon geometrically by a 1-ended group have no cut-points? does the converse of Proposition 7.1 (Prop. VII) hold, i.e. does a group acting geometrically on a $\mathfrak{ccc}$-

bicombable space X split as an amalgamated product over a 2-ended subgroup when the boundary of X has a local cut-point?

Q5. Describe the topology of boundaries of some examples of $\mathfrak{ccc}$-bicombable spaces acted upon in a controlled way by a group. For example, the C(4)–T(4) small cancellation groups and the FC-type Artin groups are Helly groups [Cha+24; HO21b], hence they act geometrically on $\mathfrak{ccc}$-bicombable spaces (see Corollary II(ii)), which provides these classes of groups with the first notion of boundary (of an associated space) known for them.

B Complete characterisations of hyperbolic Coxeter groups with Sierpiński curve boundary and with Menger curve boundary

This part of the thesis is based on the paper [DKŚ24] by the author of this thesis, Michael Kapovich and Jacek Świątkowski, sharing the title with this part, which has just appeared as an online-first article in Fundamenta Mathematicae. See the introduction to the thesis for more details.

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0. Introduction

0.1. Overview and context

It is a classical and widely open problem to characterise those word hyperbolic groups whose Gromov boundary is homeomorphic to a given topological space. The complete answers (for non-elementary hyperbolic groups) are known only for the Cantor set (virtually free groups) and for the circle S^1 (cocompact Fuchsian groups). For the sphere S^2 the expected answer is known as Cannon's Conjecture, and it remains open. Some partial answers are known in the restricted frameworks. For example, Cannon's conjecture is known to be true for Coxeter groups (we discuss this issue with more details in Subsection 1.4). In this paper we deal with spaces known as the Sierpiński curve and the Menger curve, providing complete characterisations of word hyperbolic Coxeter groups for which these spaces appear as the Gromov boundaries.

Some partial results in this direction have been presented quite recently by several authors. For example, P. Dani, M. Haulmark and G. Walsh in [DHW23] have shown that for a word hyperbolic right-angled Coxeter group W whose nerve L is 1-dimensional, ∂W is homeomorphic to the Menger curve iff L is *unseparable* (i.e. connected, with no separating vertex and no separating pair of non-adjacent vertices) and non-planar. The third author of the present paper, in [Świ16], characterised those word hyperbolic Coxeter groups with Sierpiński curve boundary whose nerves are planar complexes. The first author in [Dan22] provided a sufficient condition for the nerve of a word hyperbolic right-angled Coxeter group W, which can be applied to nerves of arbitrary dimension, under which the Gromov boundary ∂W is the Menger curve.

This paper resulted from an observation (by the second author) that some results of M. Bourdon and B. Kleiner from [BK13] can be applied to obtain the complete characterisations, as presented below.

0.2. Results

Before stating our main result we need to recall some terminology and notation appearing in its statement. The *nerve* of a Coxeter system (W, S) is the simplicial complex $L = L(W, S)$ whose vertex set is identified with S and whose simplices correspond to those subsets $T \subset S$ for which the special subgroup W_T is finite. The *labelled nerve* $L^\bullet$ of (W, S) is the nerve L in which the edges are equipped with labels in such a way that any edge $[s, t]$ has label equal to the exponent m_{st} from the standard presentation associated to (W, S) (equivalently, m_{st} is the appropriate entry of the Coxeter matrix of the system (W, S)). Obviously, the labelled nerve of a Coxeter system carries the same information as its Coxeter matrix. Note that the labelled nerve of the direct product of two Coxeter systems is the simplicial join of the nerves of the two factors, where the labels at edges of the joined complexes are preserved, and the labels at all 'connecting' edges (i.e. edges having endpoints in both joined complexes) are equal to 2. We call such a labelled nerve *the labelled join* of the labelled nerves of the two factors. A Coxeter system is called *indecomposable* if it cannot be expressed as a direct product of non-trivial Coxeter systems. Observe that a Coxeter system is indecomposable iff its labelled nerve cannot be expressed as a labelled join of two non-trivial labelled complexes.

We use the convention of speaking of topological or simplicial properties of labelled nerves as of the properties of the corresponding underlying unlabelled nerves. The labelled nerve of a Coxeter system is *unseparable* if it is connected, has no separating simplex, no separating pair of non-adjacent vertices, and no separating *labelled suspension* (i.e. a full subcomplex which is the labelled join of a simplex and a doubleton). The concept of unseparability is useful because of the following characterisation of non-existence of a splitting along a finite or a 2-ended subgroup in a Coxeter group, due to Mihalik and Tschantz [MT09]: *the group W in a Coxeter system (W, S) has no non-trivial splitting along a finite or a 2-ended subgroup iff its labelled nerve is unseparable* (see Subsection 1.2

for more details).

Given a finite simplicial complex K we define its *puncture-respecting cohomological dimension*, denoted as $\mathrm{pcd}(K)$, by the formula

$$\mathrm{pcd}(K) := \max\{n : \overline{H}^n(K) \neq 0 \text{ or } \overline{H}^n(K \setminus \sigma) \neq 0 \text{ for some } \sigma \in \mathcal{S}(K)\},$$

where $\mathcal{S}(K)$ is the family of all closed simplices of K. This concept is useful for us due to its role in a formula (by M. Davis) for the virtual cohomological dimension of a Coxeter group, see Proposition 1.3 below, and its proof.

A *3-cycle* is a triangulation of the circle S^1 consisting of precisely 3 edges.

Our main result is the following.

Theorem I. *Let (W, S) be an indecomposable Coxeter system such that W is infinite word hyperbolic, and let $L^\bullet$ be its labelled nerve.*

(i) The Gromov boundary ∂W is homeomorphic to the Sierpiński curve iff $L^\bullet$ is unseparable, planar (in particular, not a triangulation of S^2), and not a 3-cycle.

(ii) The Gromov boundary ∂W is homeomorphic to the Menger curve iff $L^\bullet$ is unseparable, $\mathrm{pcd}(L^\bullet) = 1$, and $L^\bullet$ is not planar.

Remark II. (i) Recall that W is infinite iff its nerve is not a simplex. Recall also that word hyperbolicity of W has been characterised by G. Moussong (see [Mou88], or Theorem 12.6.1 in [Dav08]) as follows: W is word hyperbolic iff it has no affine special subgroup of rank ≥ 3, and no special subgroup which decomposes as the direct product of two infinite special subgroups.

(ii) One of the consequences of the above Moussong's characterisation of word hyperbolicity is as follows. A word hyperbolic infinite Coxeter group decomposes (uniquely) into the direct product of an infinite indecomposable special subgroup (which is also word hyperbolic) and a finite special subgroup (possibly trivial). This allows to extend Theorem 0.1 in the obvious way to Coxeter systems (W, S) which are not necessarily indecomposable. Namely, conditions for the nerve $L^\bullet$ have to be satisfied up to the labelled join with a simplex.

(iii) The above two remarks show that Theorem 0.1 actually yields a complete characterisation (in terms of Coxeter matrices or labelled nerves) of those Coxeter systems (W, S) for which W is word hyperbolic and its Gromov boundary ∂W is homeomorphic to the Sierpiński curve or to the Menger curve. We skip the straightforward details of such characterisations.

Plan of the paper. In Section 1 we collect various (rather numerous) preparatory results, and in Section 2 we provide the main line of the argument of the proof of Theorem I (which is relatively short).

More precisely, here is the structure of Section 1. In Subsection 1.1 we recall the famous topological characterisations of the Sierpiński curve and of the Menger curve, due to Whyburn [Why58] and to Anderson [And58], respectively. In Subsection 1.2 we present a complete characterisation (in terms of labelled nerves) of those word hyperbolic Coxeter groups whose Gromov boundary is connected and has no local cut-points. As we explain, this characterisation is a more or less direct consequence of the results of Bowditch [Bow98], Davis [Dav98; Dav08], and Mihalik and Tschantz [MT09]. In Subsection 1.3 we present a useful formula for the topological dimension of the Gromov boundary of a word hyperbolic Coxeter group, which is due to Davis [Dav98] and Bestvina and Mess [BM91]. In Subsection 1.4 we recall a result of Bourdon and Kleiner [BK13], which confirms the Cannon's conjecture in the framework of word hyperbolic Coxeter groups. In Subsection 1.6 we discuss another result, which is implicit in the paper [BK13] by Bourdon and Kleiner, namely the fact that if the Gromov boundary of an indecomposable word hyperbolic Coxeter group is the Sierpiński curve then the nerve of the corresponding Coxeter system is a planar simplicial complex. Since the arguments for this fact provided in [BK13] are extremely sketchy, we include an extended exposition of its proof. In particular, in this exposition we refer to some auxiliary result from combinatorial group theory, which we state and prove in Subsection 1.5, and for which we couldn't find an appropriate reference in the literature.

The proof of Theorem I provided in Section 2 is split into separate parts concerning the Menger curve and the Sierpiński curve. It uses all the preparatory results from Section 1.

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1. Preliminaries and preparations

In this section we collect various useful results from the literature (or some more or less direct consequences of such results), and few other preparatory observations. We will refer to all these results in our main arguments in Section 2.

1.1. Characterisations of the Sierpiński curve and of the Menger curve

By a result of Whyburn [Why58], the Sierpiński curve is the unique metrisable topological space which is compact, connected, locally connected, 1-dimensional, without local cut-points and planar. A somewhat similar result of Anderson [And58] characterises the Menger curve as the unique compact metrisable space which is connected, locally connected, 1-dimensional, has no local cut-points, and is nowhere planar (nowhere planarity means that no open subset of the space is planar).

By referring to the above characterisations, the second author and B. Kleiner made in their paper [KK00] the following observation.

Proposition 1.1 (M. Kapovich and B. Kleiner [KK00]). *Let G be a word hyperbolic group, and suppose that its Gromov boundary ∂G is connected, 1-dimensional, and has no local cut-points. Then ∂G is homeomorphic either to the Sierpiński curve or to the Menger curve.*

1.2. Connectedness and non-existence of local cut-points in the Gromov boundary ∂W

It is a well known fact that once a hyperbolic group is 1-ended then its Gromov boundary is not only connected, but also locally connected (see e.g. Theorem 7.2 in [KB02]). This allows to discuss existence of local cut-points in the boundary. As far as this issue, we have the following observation, which probably belongs to folklore.

Proposition 1.2. *Let (W, S) be a Coxeter system, and let $L^\bullet$ be its labelled nerve. Suppose also that the group W is infinite and word hyperbolic. Then the Gromov boundary ∂W is connected and has no local cut-points iff $L^\bullet$ is unseparable and not a 3-cycle.*

Proof. Step 1. Since connectedness of the boundary ∂W is equivalent to 1-endedness of W, by Theorem 8.7.2 in [Dav08] we get that ∂W is connected iff the nerve L is connected and has no separating simplex.

Step 2. By Theorem 8.7.3 in [Dav08], a Coxeter group is 2-ended iff it decomposes as the direct product of its infinite dihedral special subgroup and its finite (possibly trivial) special subgroup. Equivalently, a Coxeter group is 2-ended iff its labelled nerve is either a doubleton or a labelled suspension (as defined in the introduction).

As a consequence of the above, if the group W is 1-ended, non-existence of a separating pair of non-adjacent vertices and of a separating labelled suspension (in the labelled nerve

$L^\bullet$) means exactly that W does not visually split (in the sense of the paper [MT09] by Mihalik and Tschantz) over a 2-ended subgroup. More precisely, this means that W cannot be expressed as an essential free product of its two special subgroups, amalgamated along a 2-ended special subgroup. It follows from the main result of the same paper [MT09] that non-existence of a separating pair of non-adjacent vertices and of a separating labelled suspension in $L^\bullet$ is equivalent to the fact that W does not split along any 2-ended subgroup.

Step 3. By a result of Bowditch [Bow98], the Gromov boundary ∂G of a 1-ended hyperbolic group G has no local cut-point iff G has no splitting along a 2-ended subgroup and is not a cocompact Fuchsian group. By a result of Davis (see Theorem B in [Dav98] or Theorem 10.9.2 in [Dav08]), a Coxeter group is a cocompact Fuchsian group iff its nerve is either a triangulation of S^1 or the group splits as the direct product of a special subgroup with the nerve S^1, and another special subgroup, which is finite. It follows from these two results, and from the conclusion of Step 2, that the Gromov boundary ∂W of a 1-ended word hyperbolic Coxeter group W has no local cut-point iff its labelled nerve $L^\bullet$ has no separating pair of non-adjacent vertices, no separating labelled suspension, and is not a 3-cycle.

Step 4. Proposition 1.2 follows by combining the observations of Steps 1 and 3. □

1.3. Topological dimension of the Gromov boundary ∂W

Recall that, given a finite simplicial complex K we have defined (in the introduction) its *puncture-respecting cohomological dimension*, denoted as $\mathrm{pcd}(K)$, by the formula

$$\mathrm{pcd}(K) := \max\{n : \overline{H}^n(K) \neq 0 \text{ or } \overline{H}^n(K \setminus \sigma) \neq 0 \text{ for some } \sigma \in \mathcal{S}(K)\},$$

where $\mathcal{S}(K)$ is the family of all closed simplices of K. The role of this concept for our considerations in this paper comes from the following observation.

Proposition 1.3. *Let (W, S) be a Coxeter system, and let L be its nerve. Suppose also that the group W is word hyperbolic. Then*

$$\dim \partial W = \mathrm{pcd}(L).$$

Proof. Denote by $\mathrm{vcd}(W)$ the virtual cohomological dimension of W. It follows from results of Mike Davis that $\mathrm{vcd}(W) = \mathrm{pcd}(L) + 1$ (see Corollary 8.5.5 in [Dav08]). On the other hand, by the result of M. Bestvina and G. Mess [BM91], we have $\mathrm{vcd}(W) = \dim \partial W + 1$, hence the proposition. □

1.4. Cannon's conjecture for Coxeter groups

The following result has been proved using quite advanced methods by M. Bourdon and B. Kleiner in [BK13], and its short proof as presented below (indicated by M. Davis) has been also outlined in the same paper. We include this short proof for completeness (since our statement, being convenient for our applications, is not identical to that in [BK13]), and for reader's convenience.

Proposition 1.4. *Let (W, S) be an indecomposable Coxeter system, and let L be its nerve. Suppose also that the group W is word hyperbolic. Then the following conditions are equivalent:*

(i) $\partial W \cong S^2$,

(ii) L *is a triangulation of* S^2,

(iii) W *acts properly discontinuously and cocompactly, by isometries, as a reflection group, on the hyperbolic space* $\mathbb{H}^3$.

Proof. We justify the implications (i)⇒(ii)⇒(iii)⇒(i).

The (i)⇒(ii) implication. By result of M. Bestvina and G. Mess (Corollary 1.3(c) in [BM91]), if $\partial W \cong S^2$ then W is a virtual Poincaré duality group of dimension 3. By result of M. Davis (Theorem 10.9.2 in [Dav08]), the nerve L is then a triangulation of S^2 (here we use the assumption of indecomposability).

The (ii)⇒(iii) implication. This implication follows by applying Andreev's theorem (see [And70], or Theorem 6.10.2 in [Dav08]) to the dual polyhedron of the triangulation.

The (iii)⇒(i) implication. By the assumptions on W in condition 3, we obviously have $\partial W = \partial \mathbb{H}^3$, and the implication follows from the fact that $\partial \mathbb{H}^3 \cong S^2$. □

For the later arguments of this paper we only need the implication (i)⇒(ii).

1.5. An observation from combinatorial group theory

Let Γ be an arbitrary group and H_i for $1 \leq i \leq n$ be a collection of its (not necessarily pairwise distinct) subgroups. In this subsection we describe two group operations associated to this data, and discuss the relationship between the groups obtained by these operations. This observation (Lemma 1.7 below) will be useful in the argument in Subsection 1.6.

In the next definition we describe the first of the two operations, which the second author and B. Kleiner call the *double* of Γ with respect to the tuple (H_i) (see [KK00]).

Definition 1.5. Given a group Γ and a finite tuple of its subgroups (H_i), the *double* $\Gamma \circledast \Gamma$ is the fundamental group $\pi_1\mathcal{G}$ of the graph of groups $\mathcal{G}$ described as follows. The underlying graph of $\mathcal{G}$ consists of two vertices v and v' and n edges $e_1, \ldots, e_n$ each of which has both v and v' as its endpoints. The vertex groups at v and v' are both identified with Γ while the edge group at any edge e_i is identified with H_i. The structure homomorphisms are all taken to be the inclusions.

Let $\Gamma = \langle S|R\rangle$, and let $\Gamma' = \langle S'|R'\rangle$ be a second copy of Γ (given by the same presentation). Denote by $\mathcal{W}_{H_i}$ the set of words over $S \cup S^{-1}$ that represent elements of the subgroup H_i and for a word w over $S \cup S^{-1}$ let w' be the word over $S' \cup S'^{-1}$ obtained from w by substituting each letter with its counterpart from $S' \cup S'^{-1}$. Note that (e.g. by Definition 7.3 in [DD89]), the double $\Gamma \circledast \Gamma$ can be also described as follows. Consider an auxiliary group $P = P(\Gamma, (H_i))$ given by the presentation

$$\left\langle S \sqcup S' \sqcup \{u_i : 1 \leq i \leq n\} \,\middle|\, R \cup R' \cup \{h_i u_i = u_i h_i' : 1 \leq i \leq n, h_i \in \mathcal{W}_{H_i}\}\right\rangle.$$

Then $\Gamma \circledast \Gamma$ is a subgroup of P consisting of all elements p such that there exists an expression $p = w_0 u_{i_1} w_1 u_{i_2}^{-1} w_2 \ldots w_{2m-1} u_{i_{2m}}^{-1} w_{2m}$ for some $m \geq 0$, $1 \leq i_k \leq n$ and words w_k over $S \cup S^{-1}$ and $S' \cup S'^{-1}$ for even and odd k respectively.

The second of the group operations is given in the following.

Definition 1.6. Given a group $\Gamma = \langle S|R\rangle$ and a finite tuple of its subgroups (H_i), the *mirror double* $\widetilde{\Gamma}$ of the group Γ with respect to the tuple (H_i), is the group given by the presentation

$$\begin{aligned}\widetilde{\Gamma} :=& \big\langle S \sqcup \{s_i : 1 \leq i \leq n\} \,\big| \\ & R \cup \{s_i^2 = 1 : 1 \leq i \leq n\} \cup \{h_i s_i = s_i h_i : 1 \leq i \leq n, h_i \in \mathcal{W}_{H_i}\}\big\rangle.\end{aligned}$$

Observe that the mirror double is (up to isomorphism) independent of the presentation of Γ used in the definition above.

Lemma 1.7. *For each group Γ and any finite tuple of its subgroups (H_i) the double $\Gamma \circledast \Gamma$ is isomorphic to an index 2 subgroup of the mirror double $\widetilde{\Gamma}$.*

Remark. The concepts of a double $\Gamma \circledast \Gamma$ and a mirror double $\widetilde{\Gamma}$ are well known e.g. in the context of compact hyperbolic manifolds, M, with non-empty totally geodesic boundary ∂M. If we take $\Gamma = \pi_1 M$, and if subgroups $H_i < \Gamma$ correspond to the fundamental groups of the boundary components, the double $\Gamma \circledast \Gamma$ is the fundamental group of the double DM of the manifold M along ∂M. In the same situation, the mirror double $\widetilde{\Gamma}$ corresponds to the fundamental group of the orbifold $\mathcal{O}_M$ with the underlying space M, in which the local groups at the boundary are the groups of order 2 representing geometrically local reflections. Since the double DM is obviously a degree 2 covering of the orbifold $\mathcal{O}_M$ (in the orbifold sense), the assertion of Lemma 1.7 is obvious in this situation. The full statement of Lemma 1.7 is just a group theoretic extension of that observation (which could be also given a geometrical sense).

Proof. Consider the homomorphism $\rho\colon P \to \widetilde{\Gamma}$ given by $\rho(s) = \rho(s') = s$ for each $s \in S$, and $\rho(u_i) = s_i$ for each $1 \leq i \leq n$. Consider also the homomorphism $\sigma\colon \widetilde{\Gamma} \to \mathbb{Z}_2$ given by $\sigma(s) = 0$ for $s \in S$, and $\sigma(s_i) = 1$ for $1 \leq i \leq n$. It suffices to show that ρ restricts to an isomorphism between $\Gamma \circledast \Gamma$ and $\ker \sigma$, which is an index 2 subgroup of $\widetilde{\Gamma}$. It is easy to check that $\rho(\Gamma \circledast \Gamma) = \ker \sigma$, so it remains to show that $\rho|_{\Gamma\circledast\Gamma}$ is injective. To this end we introduce the following lift function $\ell\colon \ker \sigma \to \Gamma \circledast \Gamma$. For an element $\xi \in \ker \sigma$, and for its any expression by a word of the form $w_0 s_{i_1}^{\epsilon_1} w_1 s_{i_2}^{\epsilon_2} \cdot \ldots \cdot w_{2m-1} s_{i_{2m}}^{\epsilon_{2m}} w_{2m}$ for some (possibly empty) words w_i over the alphabet $S \cup S^{-1}$, $\epsilon_j \in \{-1, 1\}$ and for $1 \leq i_j \leq n$, put $\ell(\xi) := w_0 u_{i_1} w_1' u_{i_2}^{-1} \cdot \ldots \cdot w_{2m-1}' u_{i_{2m}}^{-1} w_{2m}$. The map ℓ is well defined, since it is easy to check that for each word

$$U = w_0 s_{i_1}^{\epsilon_1} w_1 s_{i_2}^{\epsilon_2} \cdot \ldots \cdot w_{2m-1} s_{i_{2m}}^{\epsilon_{2m}} w_{2m},$$

and for each elementary operation consisting of inserting at an arbitrary place in U (or deleting) a subword of the form $a^{-1}a$ for some letter a, or a relator (in $\widetilde{\Gamma}$) or inverse of such, resulting in the word $\hat{U}$, the words representing $\ell(U)$ and $\ell(\hat{U})$ in the definition of ℓ differ by an analogous elementary operation (in P). Moreover, since we then clearly have that $\ell \circ \rho|_{\Gamma\circledast\Gamma} = \mathrm{id}_{\Gamma\circledast\Gamma}$, we conclude that $\rho|_{\Gamma\circledast\Gamma}$ is injective, hence the lemma. $\square$

1.6. Planarity of nerves

We recall the following rather easy observation from the paper [Świ16] written by the third author of the present paper.

Lemma 1.8 (J. Świątkowski, Lemma 1.3 in [Świ16]). *If the nerve L of a word hyperbolic Coxeter group W is a planar complex then the Gromov boundary ∂W is a planar topological space.*

The converse implication is not true in general [DHW23], but it does hold in an important special case. This is the contents of the next result which appears implicitly as Corollary 7.5 in [BK13]. The proof given below is an expansion of a rather sketchy argument provided in [BK13].

Proposition 1.9. *Let (W, S) be an indecomposable Coxeter system such that the group W is word hyperbolic. If the Gromov boundary ∂W is homeomorphic to the Sierpiński curve then the nerve L of the system (W, S) is a planar simplicial complex.*

Proof. We will embed the group W, as a special subgroup, in some larger indecomposable and word hyperbolic Coxeter group $\widetilde{W}$ such that $\partial\widetilde{W} \cong S^2$. The assertion will follow then from the implication (i)⇒(ii) in Proposition 1.4.

We start by recalling some facts established in the paper [KK00] by the second author and B. Kleiner. First, the Sierpiński curve contains the family of topologically well distinguished pairwise disjoint subsets homeomorphic to S^1, called *peripheral circles*. Moreover, in its action on ∂W the group W maps peripheral circles to peripheral circles. A setwise stabiliser of each peripheral circle in ∂W, called a *peripheral subgroup* of W, is a quasi-convex subgroup of W for which the circle is its limit set, and consequently each such stabiliser is a cocompact Fuchsian group. The action of W on the family of peripheral circles in ∂W has finitely many orbits, and thus we have finitely many conjugacy classes of peripheral subgroups in W.

Claim. *Each peripheral subgroup of W is a conjugate of some special subgroup of W.*

To prove this claim we need some terminology and notation as in Section 5.1 in [BK13]. For a generator $s \in S$, the *wall* M_s is the set of setwise s-stabilised open edges of $\mathrm{Cay}(W,S)$ (the Cayley graph of W with respect to the set of generators S). Then $\mathrm{Cay}(W,S) \setminus M_s$ consists of two connected components $H_-(M_s)$ and $H_+(M_s)$. For a generator $s \in S$ and for an arbitrary element $g \in W$ we consider the reflection $r := gsg^{-1}$, its wall $M_r := gM_s$ and components $H_-(M_r)$ and $H_+(M_r)$ of $\mathrm{Cay}(W,S)\setminus M_r$. The components are closed and convex subsets of $\mathrm{Cay}(W,S)$ and $\partial H_-(M_r) \cup \partial H_+(M_r) = \partial W$, $\partial H_-(M_r) \cap \partial H_+(M_r) = \partial M_r$ and r pointwise stabilises ∂M_r.

Proof. In view of Definition 5.4 and Theorem 5.5 in [BK13] it suffices to show that for each peripheral circle F and each reflection r such that $\partial H_-(M_r)\cap F$ and $\partial H_+(M_r)\cap F$ are non-empty, it holds that F is setwise stabilised by r. Since $(\partial H_-(M_r)\cap F)\cup(\partial H_+(M_r)\cap F) = \partial W\cap F = F$, by connectedness of $F \cong S^1$, we have that $\emptyset \neq (H_-(\partial M_r)\cap F)\cap(H_+(\partial M_r)\cap F) = \partial M_r \cap F$. Since ∂M_r is pointwise stabilised by r, $rF \cap F \neq \emptyset$, and, finally, $rF = F$ by the fact that each element of W maps peripheral circles to peripheral circles.

Coming back to the proof of Proposition 1.9, denote by $H_i : 1 \leq i \leq n$ a set of representatives of the conjugacy classes of peripheral subgroups of W consisting of special subgroups of W. For each $1 \leq i \leq n$, denote by L_i the nerve of H_i, and view it as a subcomplex of the nerve L of W. We will discuss below the double $W \circledast W$ and the mirror double $\widetilde{W}$ of W with respect to the tuple (H_i) (see Subsection 1.5). As it is shown in [KK00], the double $W \circledast W$ is a hyperbolic group and its Gromov boundary is homeomorphic to S^2. Observe also that the mirror double $\widetilde{W}$ is (isomorphic to) a Coxeter group with nerve $\widetilde{L}$ obtained from the nerve L of W by adding a simplicial cone over each of the subcomplexes L_i. Moreover, since each H_i is a proper special subgroup of W, indecomposability of W implies indecomposability of $\widetilde{W}$. By Lemma 1.7, the group $\widetilde{W}$ contains $W \circledast W$ as a subgroup of index 2, and hence it is also word hyperbolic and its Gromov boundary is homeomorphic to S^2. By Proposition 1.4, $\widetilde{L}$ is then a triangulation of S^2. Since L is clearly a proper subcomplex of $\widetilde{L}$, it is necessarily planar, which completes the proof of Proposition 1.9. $\square$

2. Proof of the main theorem

2.1. Sierpiński curve boundary

In this rather short subsection we prove part (i) of Theorem I.

THE $\Longrightarrow$ IMPLICATION. Suppose that ∂W is homeomorphic to the Sierpiński curve. Then, in view of the fact that the Sierpiński curve is connected and has no local cut-points, it follows from Proposition 1.2 that $L^\bullet$ is unseparable and not a 3-cycle. Moreover, by Proposition 1.9, L is then a planar simplicial complex, which completes the proof.

THE $\Longleftarrow$ IMPLICATION. As any Gromov boundary of a hyperbolic group, ∂W is a compact metrisable space. Since L is planar, it follows from Lemma 1.8 that ∂W is a planar space. Since $L^\bullet$ is unseparable and not a 3-cycle, it follows from Proposition 1.2 that ∂W is connected, locally connected, and has no local cut-point. Finally, it is not hard to see that since L is planar, connected, has no separating simplex, and does not coincide with a single simplex, its puncture-respecting cohomological dimension $\mathrm{pcd}(L)$ is equal to 1. Consequently, due to Proposition 1.3, ∂W has topological dimension 1. Thus, by Whyburn's characterisation recalled in Subsection 1.1, ∂W is homeomorphic to the Sierpiński curve, as required.

2.2. Menger curve boundary

We now pass to the proof of part (ii) of Theorem I.

THE $\Longrightarrow$ IMPLICATION. Suppose that ∂W is homeomorphic to the Menger curve. Then, in view of the fact that the Menger curve is connected and has no local cut-points, it follows from Proposition 1.2 that $L^\bullet$ is unseparable. Since the Menger curve has topological dimension 1, it follows from Proposition 1.3 that $\mathrm{pcd}(L) = 1$. Since the Menger curve is not planar, it follows from Lemma 1.8 that L is also not planar, and this completes the proof of the first implication.

THE $\Longleftarrow$ IMPLICATION. The boundary ∂W is obviously a compact metrisable space. Since $L^\bullet$ is not planar, not a 3-cycle, and since $L^\bullet$ is unseparable, it follows from Proposition 1.2 that ∂W is connected, locally connected, and has no local cut-point. Since $\mathrm{pcd}(L) = 1$, it follows that ∂W has topological dimension 1. In view of the above properties, it follows from Proposition 1.1 that ∂W is homeomorphic either to the Sierpiński curve or to the Menger curve. However, since L is not planar, it follows from Proposition 1.9 that ∂W cannot be homeomorphic to the Sierpiński curve. Consequently, it must be homeomorphic to the Menger curve, as required.

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