EXAM Graphical Models. TIME: 45 min.

ALTERNATIVE WAY OF GETTING THE FINAL NOTE WITHOUT COMPUTER LAB REPORT:
PRESENT EXERCISES FROM PARTS 3,4,5 + EXAM BY EMAIL

Let X be a centered Gaussian vector of dimension 3 given by $X = \left(X_1, X_2, X_3\right)^T \sim N(0, \Sigma_X)$ with covariance matrix $\Sigma_X = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ and precision matrix $K_X = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.

1. What is the relation between matrices Σ_X and K_X?
2. Are there independent components X_i? If yes, which ones?
3. Are there components X_i conditionally independent, knowing the others? If yes, which ones? What can we deduce on the prediction of X_1, if one knows X_2 and X_3?
4. Draw the dependance graph $\mathcal{G}$ of X.
5. Determine the marginal law of $(X_2, X_3)^T$.
6. Determine the conditional law of $(X_2, X_3)^T | X_1 = u$ and the conditional correlation $\rho_{X_2, X_3 | X_1 = u}$.
7. One knows that the random vector Y belongs to the graphical Gaussian model governed by the graph $\mathcal{G}$. One does not know the covariance matrix Σ_Y of Y.

   We have a sample of size $n = 5$ of Y and one computes the sample (empirical) covariance matrix $\tilde{\Sigma}_Y = \begin{pmatrix} 2 & 1 & 0.9 \\ 1 & 1 & 1 \\ 0.9 & 1 & 2 \end{pmatrix}$.

   Give the ML Estimator (MLE) Σ_Y and the MLE of the precision matrix K_Y of Y.
8. Is the graph $\mathcal{G}$ complete? Decomposable? Give its decomposition into cliques.
9. Give an example of a non-decomposable graph.

Some formulas from the lectures. *Let X be a Gaussian vector $N(\xi, \Sigma)$ in $\mathbf{R}^d$ with Σ invertible.*
One partitions $X = \begin{pmatrix} X_A \\ X_B \end{pmatrix}$ into sub-vectors $X_A \in \mathbf{R}^r$ and $X_B \in \mathbf{R}^s$, where $r + s = d$.
One partitions $\xi = \begin{pmatrix} \xi_A \\ \xi_B \end{pmatrix}$, $\Sigma = \begin{pmatrix} \Sigma_{AA} & \Sigma_{AB} \\ \Sigma_{BA} & \Sigma_{BB} \end{pmatrix}$, $K = \begin{pmatrix} K_{AA} & K_{AB} \\ K_{BA} & K_{BB} \end{pmatrix}$ into blocs $\begin{pmatrix} r \times r & r \times s \\ s \times r & s \times s \end{pmatrix}$.

The **conditional law** $X_A | \ (X_B = x_B) \ \sim N(\xi_{A|B}, \Sigma_{A|B})$ *where*

$$\xi_{A|B} = \xi_A + \Sigma_{AB}\Sigma_{BB}^{-1}(x_B - \xi_B) \text{ and } \Sigma_{A|B} = K_{AA}^{-1}.$$

The **conditional correlation** $\rho_{lm|V\setminus\{l,m\}} = -\tilde{\kappa}_{lm} = -\dfrac{\kappa_{lm}}{\sqrt{\kappa_{ll}}\sqrt{\kappa_{mm}}}$.

Maximum Likelihood Equation : $\pi_{\mathcal{G}}(\hat{K}^{-1}) = \pi_{\mathcal{G}}(\tilde{\Sigma})$, where $\tilde{\Sigma}$ is the sample covariance.