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Estimates of integral kernels

PhD thesis

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Abstract

The dissertation grew out from attempts to better understand the estimates of Schrödinger perturbations of transition densities given in [1].

Schrödinger perturbation consists in adding to a given operator, say the Laplacian Δ, an operator of multiplication by a function, say q. Local integral smallness of the function q, formulated by Kato-type conditions [5, 9], played an important role in these considerations. The heat kernel estimates for $\Delta^{\alpha/2}+q$, in fact Schrödinger-type perturbations of general transition densities were then studied in [1] under the following integral condition on q,

$$\int_s^t \int_X p(s,x,u,z)|q(u,z)|p(u,z,t,y)dzdu \leq [\eta + \gamma(t-s)]p(s,x,t,y).$$

Here p is a finite positive jointly measurable transition density, γ and η are fixed nonnegative numbers, while times $s < t$ and states x, y are arbitrary. Given the above assumption, the following explicit estimate was obtained in [1] when $\eta < 1$,

$$\tilde{p}(s,x,t,y) \leq \frac{1}{1-\eta} \exp\left(\frac{\gamma}{1-\eta}(t-s)\right)p(s,x,t,y).$$

Here $\tilde{p}$ is the Schrödinger perturbation series defined by p and q. It should be noted that the results of [1] are obtained with an essential dose of combinatorics. The combinatorics results from iterated integrations on time simplexes. As a prologue to the dissertation, we examine these integrations in Chapter 2, where we present, after [8], similarity of the above estimate to Gronwall inequality. In Chapter 2 we also indicate the role of smallness of the first nontrivial term of the perturbation series. Integration on time simplexes and the smallness condition may be considered as the leading themes of the dissertation.

Inspired by [1], further combinatorial arguments based on Stirling numbers were used in [6] to refine the above result of [1] by (1) skipping the Chapman-Kolmogorov condition on p, (2) relaxing the assumptions on q, and (3) strengthening the estimate. Namely, if $0 < \eta < 1$ and $Q \geq 0$ is superadditive, then the smallness condition

$$\int_s^t \int_X p(s,x,u,z)q(u,z)p(u,z,t,y)dzdu \leq [\eta + Q(s,t)]p(s,x,t,y), \qquad (1)$$

(in short $pqp \leq [\eta + Q(s,t)]p$) leads to the main estimate of [6]:

$$\widetilde{p}(s,x,t,y) \leq \left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta} p(s,x,t,y)$$

(in short $pqp \leq (1-\eta)^{-1-Q(s,t)/\eta}p$), see Chapter 3. Meanwhile, a more straightforward method was proposed in [7] for *gradient* perturbations of the transition density of $\Delta^{\alpha/2}$. It was also suggested in [7, p. 321] that the technique may be applied to Schrödinger perturbations to produce the main results of [6]. In [2] we develop this observation in considerable generality: we estimate Schrödinger-type perturbations of Markovian semigroups, potential kernels, and general *forward* integral kernels on space-time by rather singular functions q. We obtain local in time and global in space comparability of the original and perturbed kernels under suitable smallness conditions on the first non-trivial term of the perturbation series.

The results of [2] are presented in Chapter 3, in particular in Theorem 3.1.5, which may be consider as one of the main results of the dissertation.

In Chapter 4 we are concerned with estimates of the resulting Schrödinger-type perturbations of integral kernels. Both the original kernel K and the perturbing kernel q are now *forward* kernels on space-time. The results are a straightforward extension of local, or Schrödinger perturbations of integral kernels from Chapter 3. In particular the resulting perturbation and the original kernel turn out to be comparable locally in time and globally in space under an integral smallness condition on the first nontrivial term of the perturbation series. We also present results for local in time and nonlocal in space perturbations, which interpolate between Chapter 3 and Chapter 4 and require a separate treatment. Our presentation in Chapter 4 closely follows preprint [4] and its earlier version [3]. While transition and potential kernels of Markov processes are our main motivation for this work, we emphasize that in what follows we do not generally impose Chapman-Kolmogorov condition on the kernels.

Our reasoning in Chapter 3 and Chapter 4 are based on suitable inductive estimates for the terms of the perturbation series. The estimates reflect interrelations of multiple integrations on time symplexes of different dimensions in the definition of the terms in the perturbations series, but the multiple integrations do not explicitely show in the arguments. In Chapter 5 we reexamine the integrations in the case of concave majorization. Our new approach works, e.g., for $Q(s,t) = (t-s)_+^\beta$ with $0 < \beta < 1$ (1), and leads to better exponents in estimates of the resulting perturbation series.

References

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