PhD THESIS REPORT ON

"CANONICAL QUOTIENTS IN MODEL THEORY"

by ADRIÁN PORTILLO FERNÁNDEZ

A hyperdefinable set is a quotient of a type definable set by a type definable equivalence relation. Elements of such a set are called hyperimaginaries. They were introduced by Hart, Kim and Pillay in 2000, mainly with reference to simple theories, and after the paper of Lascar and Pillay in 2001 they became important outside this context too. At the moment these objects are ubiquitous in model theory. Thus the project of Krzysztof Krupiński and Adrián Portillo Fernández of a systematic study of hyperdefinable sets in the frame of neostability theory is natural and even highly desirable. It should be mentioned that some key ideas already existed before their start (for example connections with continuous logic) and some of them were in a folklore form. In Chapter 2 and in Section 3.1 Adrián Portillo thoroughly describes the state of the art in this respect. Chapter 2 also collects some knowledge from different parts of model theory (and topological dynamics) which will be used in the main body of the dissertation. It is mostly presented in a nice form.

In Sections 3.1 - 3.2 a kind of "fundamentals" of neostability theory for hyperdefinable sets is built. Although some points look to be folklore, the whole collection of definitions, remarks and statements is impressive, and moreover, is presented in a nice, transparent fashion. It seems to me that Theorem 3.2.4 that in distal theories hyperimaginary sorts are also distal, is the main result of this part. In Corollary 3.2.5 it is deduced that in a distal theory, a hyperdefinable set X/E is stable if and only if E is bounded. In particular for a group G type definable in a distal theory the smallest type definable subgroup G^{st} with stable quotient G/G^{st} coincides wih the component G^{00}. This confirms a conjecture of Haskel and Pillay. The remaining part of Chapter 3 is entirely devoted to the group theoretic issues connected with these notions.

Section 3.3 gives a concrete example of a definable group G in an NIP theory T with $G = G^{00} \neq G^{st} \neq G^{st,0}$. This answers a question of Haskel and Pillay. The arguments used in the proof are in the traditional style of model-theoretic algebra.

Section 3.4 contains some additional comments concerning groups G definable in NIP theories with $G^{st} \neq G^{st,0}$. It consists of one general observation on consequences of $G^{st} = G^{st,0}$ and two observations concerning the possibility of forcing some additional properties of the group together with $G^{st} \neq G^{st,0}$". The middle one (Proposition 3.4.2.) is characterized as it "yields the whole class of examples where $G^{00} \neq G^{st} \neq G^{st,0}$". I would say that it yields *some* class of examples... (and this class is reasonable enough).

It is proved by Haskel and Pillay that under NIP, for a small A the group G^{st}_A (the smallest A-type definable subgroup such that the quotient G/G^{st}_A is stable) is 0-type-definable and normal. This is why we can introduce G^{st} as above without reference to parameters. Chapter 4 of the thesis concentrates on possible generalizations of this fact in the situation of the absence of the group structure. The main result of this section (according to the author, even of the whole thesis) states that this is true under reasonable assumptions. Namely, having a pair of two monster models $\mathfrak{C} \prec \mathfrak{C}'$ of a NIP theory T such that $\mathfrak{C}$ is small in $\mathfrak{C}'$ but is sufficiently saturated, assume that $p(x)$ is an A-invariant type over $\mathfrak{C}$ for a small $A \subset \mathfrak{C}$. Then, there exists a finest equivalence relation E^{st} on $p(\mathfrak{C}')$ which is relatively type-definable over a small (relative to $\mathfrak{C}$) set of parameters of $\mathfrak{C}$ and with stable quotient $p(\mathfrak{C}')/E^{st}$. The proof of this theorem is difficult. It uses some tricks which are new in this context.

Chapter 4 contains two examples where E^{st} is computed. At first sight they can look casual, but it is not so. Example 2 is exactly the theory T which gave a solution to a question of Haskel and Pillay in Section 3.3. Example 1 can be viewed as a warm up before Example 2. The author distinguishes this chapter as the main one. I think that it is the hardest part of the thesis!

Dependent theories (or NIP theories) is a major object of neostability theory. Thus it is a principal question how to develop the corresponding fundamentals for hyperimaginary sorts. Adrián Portillo studies this issue in Chapter 5 in a more general case of n-dependent theories. Using the recipe of Section 3.1 (in the case of stability) he defines n-dependent hyperdefinable sets and studies them by tools taken from continuous logic. Namely, as an intermediate step n-dependent continuous theories are characterized. After the previous work made in Section 3.1 this approach looks very natural and expected.

On the other hand there is a surprising point in this chapter. As far as I can judge the original idea of the Adrián's approach was based on adaptation of the paper of Chernikov, Palacin and Takeuchi (2014) were n-dependence theory in the first-order logic is developed. After discovering an incorrect place in it Adrián decided to start with adaptation of some work of Scow (see Section 5.1) and then apply it in the corresponding place. Characterizing the whole chapter I think that it is technical, solid, but predictable. It should be added here that again this material looks as a useful tool in future model theoretic investigations.

In Section 6.4 (the last one) Adrián Portillo observes that a hyperdefinable set X/E is stable if and only if the corresponding $\mathsf{Aut}(\mathfrak{C})$-flow on the space of types $S_{X/E}(\mathfrak{C})$ is weakly almost periodic (WAP). This extends the classical result of Ben Yaacov and Tsankov. He also observes that X/E is NIP if and only if this flow is tame (extending results of Chernikov, Simon, Ibarlucia and Khanaki). These observations naturally complement Chapters 3 and 5 of the thesis. On the other hand $E^{st}_{\emptyset}$, the $\emptyset$-type definable version of E^{st} from Chapter 4, induces a closed equivalence relation $\tilde{E}^{st}_{\emptyset}$ on $S_X(\mathfrak{C})$. In a similar fashion one obtains $\tilde{E}^{NIP}_{\emptyset}$ on this space of types. It turns out that these equivalence relations do not coincide with the finest closed $\mathsf{Aut}(\mathfrak{C})$-invariant equivalence relation on $S_X(\mathfrak{C})$ such that the flow $(\mathsf{Aut}(\mathfrak{C}), S_X(\mathfrak{C})/F_{WAP})$ is WAP, denoted by F_{WAP}, and correspondingly with F_{Tame}, the finest closed $\mathsf{Aut}(\mathfrak{C})$-invariant equivalence relation on $S_X(\mathfrak{C})$ such that the flow $(\mathsf{Aut}(\mathfrak{C}), S_X(\mathfrak{C})/F_{Tame})$ is tame (easy Proposition 6.4.5). Question 6.4.6 asks if the Ellis groups of these pairs of flows coincide.

The main effort of Chapter 6 concentrates on Ellis groups of flows of this kind. The main result of the chapter, Theorem 6.2.2, gives natural conditions which guarantee absoluteness of the Ellis group (independence of changing the monster model). In Section 6.3 these conditions are verified in all cases mentioned above.

The proof of Theorem 6.2.2 is based on the very modern theory of *pattern structures* coming from papers[1] of Krupiński, Newelski, Simon and Hrushovski (in fact, the proof concerns the automorphism group of so called "Hrushovski's core"). I like this part of the thesis! It shows that Adrián is very well prepared to deep model-theoretic work.

On the other hand I also want to mention that the introduction to this chapter, i.e. Section 6.1, is not reader-friendly. For example, why not to say in the beginning of Chapter 6 that the flow $S_X(\mathfrak{C})$ is considered with respect to $\mathsf{Aut}(\mathfrak{C})$? (The notation used in this chapter is roughly the same with the different case when the group is a sort of the theory. It is interesting that even

[1] some of them are still unpublished

in Section 2, “Background”, the group is not defined. A smart reader finds it in the abstract of the thesis.) I have not found the place where morphisms and partial morphisms are defined. In the beginning of Section 6.1 Adrián Portillo informs us that this material comes from Krzysztof Krupiński course "Topological dynamics in model theory" given at University of Wrocław in 2020/2021. We can conclude that (a) it is not published and (b) it does not belong the author. I think that Adrián should be especially careful in this section. In particular, he should not leave the proof of Fact 6.1.6 to the reader.

The thesis consists of four papers. Two of them have been already published (jointly with Krzysztof Krupiński). It is clear that to a large extent these papers are included into the thesis without essential changes. As a result some places are too laconic.

My general opinion is as follows. The dissertation belongs to the model-theoretic mainstream. This research is highly desirable. Two open question formulated is a paper of other mathematicians have been answered in it. I believe that assuming that this work does not exist, sooner rather than later another model-theorist would try to investigate this topic. The candidate demonstrates good working knowledge of the subject and of some other advanced areas of mathematics (for example topological dynamics). I expect that the unpublished part of the thesis will be published in prestigious logical journals.

In my opinion the PhD thesis of Adrián Portillo Fernández satisfies all requirements of "Art.187 (ust. 1 - 3), Ustawa z dnia 20 lipca 2018 r. Prawo o szkolnictwie wyższym i nauce". I recommend proceeding to further steps of the doctor defence procedure.

I further recommend awarding the degree with distinction. It should be nominated for a prestigious doctoral dissertation award.

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