Report on the PhD thesis
On some properties of solutions to the stochastic recurrence equation by Witold Swiatkowski

Sebastian Mentemeier

July 24, 2019

On the scale $0 \dots 10$, where 5 and above means passing and 10 excellent, **I would rate this thesis with a 7.**

1 General report

1.1 Content

The thesis at hand is concerned with random d-dimensional vectors X, satisfying a *stochastic fixed point equation*

$$X \stackrel{d}{=} AX + B, \tag{1}$$

where $\stackrel{d}{=}$ denotes equality in law, and A, B are a (given) random $d \times d$-matrix and a (given) random d-dimensional vector, respectively.

Under weak assumptions, there is a unique solution (in law) to this equation, and a famous result by Kesten [2] (simplified for $d = 1$ by Goldie [1]) states that the solution is X may have regularly varying tails, i.e. for some $\alpha > 0$,

$$\lim_{t\to\infty} t^\alpha \mathbb{P}(|X| > t) \;=\; c \in (0,\infty), \tag{2}$$

even if A und B have light tails.

In the multivariate setting of Kesten [2] it is assumed that some finite product of i.i.d. copies of A has all entries strictly positive with positive probability. In a sense, this can be seen as an irreducibility assumption which in particular yields that

$$\lim_{t\to\infty} t^\alpha \mathbb{P}(X_i > t) \;=\; c_i \in (0,\infty) \quad \text{for all } 1 \le i \le d,$$

i.e., all marginals of X are regularly varying of the same order.

Fallacy of this irreducibility assumption is the main motivation of this thesis: Here, A is considered to be an upper triangular matrix with nonnegative entries with probability 1. Then any product of i.i.d. copies of A remains an upper triangular matrix. Eq. (1) still has a unique solution in law under mild assumptions, and the two main results of this thesis (Theorems 3.2.3 and 4.1.3) show that in this new setting, the marginals behave differently.

More precisely, let $A = (A_{ij})_{i,j}$ and define α_i by the condition $\mathbb{E}A_{ii}^{\alpha_i} = 1$. In **Chapter 3** and Theorem 3.2.3, it is assumed that $\alpha_i \neq \alpha_j$ for $i \neq j$. Then it holds that

$$\lim_{t\to\infty} t^{\tilde{\alpha}_i}\mathbb{P}(X_i > t) \;=\; \tilde{c}_i \in (0,\infty)$$

where $\tilde{\alpha}_i$ are defined as follows: Let $(A_n)_n$ be i.i.d. copies of A and denote $\Pi_n = A_1 \cdots A_n$. We say that i depends on j if there is $n \in \mathbb{N}$ such that $(\Pi_n)_{i,j} > 0$ with positive probability. Then

$$\tilde{\alpha}_i \;:=\; \min\{\alpha_j \,:\, i \text{ depends on } j\}. \tag{3}$$

This is what one would expect, since (possibly after iteration) Eq. (1) gives that X_i can be rewritten as a scalar combination of $\{X_j \,:\, i \text{ depends on } j\}$. For each X_j, it holds that $X_j \geq Y_j$, where Y_j satisfies

$$Y_j \;\stackrel{d}{=}\; A_{jj}Y_j + B_j.$$

It follows from the result of Goldie [1] that the tails of X_j are at least as heavy as $t^{-\alpha_j}$. Thus, (3) can be rephrased as: "the heaviest tail wins".

In **Chapter 4**, the assumption that all α_j are distinct, is removed. This means, that there can be more than one "dominant" contribution to X_i. In this setting, upper and lower bounds on the tails of X_i are derived in Theorem 4.1.3: For t sufficiently large,

$$c \cdot t^{-\tilde{\alpha}_i} \;\leq\; \mathbb{P}(X_i > t) \;\leq\; C \cdot t^{-\tilde{\alpha}_i}(\log t)^{\xi(i)}$$

for positive constants c, C and positive integer $\xi(i)$, defined precisely in Proposition 4.3.2. The author conjectures (middle of p.5) that the right hand side with ξ replaced by $\xi - 1$ would be the precise order. It is emphasized that obtaining an index $\xi(i)$ as small as possible requires most of the work in Chapter 4.

The shorter **Chapter 5** shows that if A has nonnegative entries, it can always be assumed that A is of block-upper triangular form. **Chapter 6** deals with the one-dimensional setting, providing various nontrivial sufficient conditions for the law of X to be absolutely continuous.

Chapters 1 and 2 are introductions.

1.2 Evaluation

The investigation of solutions to Eq. (1), where A is upper triangular, has started only recently. Up to now, only partial results in dimension $d = 2$ are available. The **original contribution** of this thesis is to study, for the first time, the general case ($d \geq 2$) and to give the exact tail behavior (Chapter 3) or supposingly close-to-optimal upper bounds (Chapter 4) in different setups. The arguments given require a very careful analysis of the structure of the problem, new techniques have been developed by the author. The estimates used in the proofs are intricate (see e.g. Lemma 4.3.16) and show that the author has a profound command of mathematical manipulations.

The author has provided original solutions to open scientific problems and deserves awarding a PhD. The proofs are correct and have required a lot of effort.

However, I have some critics that prevent a better grade: To judge a thesis as excellent, one would expect precise asymptotics also in Chapter 4, or at least also a lower bound that includes a logarithmic term to actually prove that the tail behavior is qualitatively different in Chapters 3 and 4, respectively.

Too little effort was made for a good presentation: The notation changes between chapters - compare Eq.s (4.1.3) and (3.1.1). The proof of Lemma 4.3.13 has 10 pages; it should have been divided into smaller results; so that the reader can understand step by step. Some a-priori discussion about the structure of the problem would have been helpful: The d-dimensional problem can be divided into possibly lower-dimensional problems, since X_i is only influenced by those coordinates, on which i depends. Thus, one could have studied from the beginning the problem in the setting, where all coordinates influence i (and neglect the others).

In several places, arguments are too complicated, simpler ones are available. See the paragraph "Further comments" below.

2 Further comments

p.9↓3 means in the third line from above on page 9, (2.3)↑2 means two lines above from Equation (2.3), p.3↑2 is the second line on p.3 from below etc.

p.4↓2 "Tail asymptotics"

p.14 The proof of Lemma 3.3.2 can be simplified as follows: Let $j_1 < \cdots < j_k$ be the collection of indices on which i depends, denote $J := \{i, j_1, \dots, j_k\}$ Then it suffices to study the reduced (lower-dimensional) equation

$$\tilde{X} \stackrel{d}{=} \tilde{A}\tilde{X} + \tilde{B},$$

with $\tilde{X} = (X_k)_{k\in J}$, $\tilde{A} = (A_{km})_{k,m\in J}$, $\tilde{B} = (B_k)_{k\in J}$. The assumption $\alpha < \tilde{\alpha}_i$ gives that $\mathbb{E}A_{kk}^{\alpha} < 1$ for all $k \in J$ and it follows that $\mathbb{E}\|\tilde{A}\|_2^{\alpha} < 1$, since the eigenvalues of a upper triangular matrix are the diagonal entries. With this, one can directly show that the vector $\tilde{X}$ has a finite moment of order α.

p.39↑2 This estimate on $\mathbb{E}X_j^{\alpha_i - \epsilon}$ is much too complicated. Using

$$\mathbb{E}X_j^{\alpha_i-\epsilon} = \int_0^{\infty} (\alpha_i - \epsilon)t^{\alpha_i-\epsilon-1}\mathbb{P}(X_j > t)\, dt$$

and the fact that $\mathbb{P}(X_j > t) = O(t^{-\alpha_i} \cdot (\log t)^{r(j)+1})$ [using here that $\alpha_i = \tilde{\alpha}_i \leq \tilde{\alpha}_j$], we immediately obtain

$$\begin{aligned}
\mathbb{E}X_j^{\alpha_i-\epsilon} &\leq 1 + C\int_1^\infty t^{-\epsilon-1}(\log t)^{r(j)+1}\,dt \\
&= 1 + C\int_0^\infty s^{r(j)+1}e^{-(1+\epsilon)s}\,e^s ds \\
&= 1 + C\int_0^\infty s^{r(j)+2-1}e^{-\epsilon s}ds \;=\; 1 + C\frac{\Gamma(r(j)+2)}{\epsilon^{r(j)+2}}.
\end{aligned}$$

In the second line, we have substituted $t = e^s$, in the third line, we have used properties of the Gamma distribution.

p. 50f It would be nice to have a "probabilistic approach" to this problem, i.e., considering a uniform distribution on all sequences and estimating (from above) the probability p_k to have k factors which are bounded by δ from above, then one could estimate

$$W_n(g, S') \leq \delta^k p_k$$

I've send a suggestion how to do this to the author.

Ch. 5 Chapter 5 is unnecessary. Its result is a direct consequence of well-known results on discrete Markov chains. Since all entries are nonnegative, A is block-upper triangular if and only if $\mathbb{E}A$ is block-upper triangular, and it is sufficient to find a change of basis for $\mathbb{E}A$. This is a deterministic matrix with nonnegative entries; and we can normalize each row such that $\mathbb{E}A$ becomes a stochastic matrix; i.e. the transition matrix of a Markov chain with state space $\{1, \ldots, d\}$. Then i, j are equivalent in the sense of Def. 5.2.4 if and only if they communicate. The blocks in Lemma 5.2.7. correspond to communication classes; the phenomenon described in Example 5.2.9. corresponds to periodicity, and validity of Kesten's assumption corresponds to irreducibility and aperiodicity.

p.83, [6] List of references should have been updated!

References

[1] Charles M. Goldie. Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.*, 1(1):126–166, 1991.

[2] Harry Kesten. Random difference equations and renewal theory for products of random matrices. *Acta Math.*, 131:207–248, 1973.