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UNIQUENESS OF THE SOLUTIONS OF $u_t - \Delta\varphi(u) = 0$ WITH INITIAL DATUM A MEASURE*

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INTRODUCTION

LET US first state the main result of this paper. Let $\varphi : [0, \infty) \to [0, \infty)$ be nondecreasing, locally Lipschitz and $\varphi(0) = 0$. Let us consider the problem

$$(\mathrm{P}) \qquad \begin{cases} u \in L^1((0, T) \times R^N) \cap L^\infty((\tau, T) \times R^N), \quad \forall \tau \in (0, T), & (1) \\ u_t - \Delta\varphi(u) = 0 \quad \text{in } \mathscr{D}'((0, T) \times R^N), & (2) \end{cases}$$

where 'in $\mathscr{D}'((0, T) \times R^N)$' means 'in the sense of distributions in $(0, T) \times R^N$'.

THEOREM 1. Let u and $\hat{u}$ be two nonnegative solutions of (P). If $N = 1$ or 2, assume also that $\varphi(u)$, $\varphi(\hat{u}) \in L^1((0, T) \times R^N)$. Then

$$\lim_{t \downarrow 0} \operatorname{ess} u(t) - \hat{u}(t) = 0 \quad \text{in } \mathscr{D}'(R^N) \tag{3}$$

implies $u = \hat{u}$.

A lot of papers have already been concerned with the uniqueness of the solutions of problem (2) especially in the particular case of the porous media equation $u_t - \Delta u^m = 0$. See, for example, Oleinik [19], Kalashnikov [12], Gilding and Peletier [10], Kamin [13] for one space variable, Vol'pert and Hudjaev [22], Sabinina [20] and finally Brézis and Crandall [6] in the general case. In the latter work which recovers most of the previous uniqueness results contained in the above, the initial value is assumed to be in $L^1(R^N)$ (*or* $L^\infty(R^N)$) and (3) holds in $L^1(R^N)$.

Here the initial value is only a finite measure which is the limit of $u(t)$ in the sense of measure (only!)—its existence is implied by (1) and (2) (see Lemma 2). This leads to a more sophisticated analysis whose main difficulty is solved by using precise properties of the potentials of the functions $u(t)$ for $N \geqslant 3$. A different proof for $N = 1, 2$ is necessary due to the non-existence of potentials; it requires $\varphi(u) \in L^1$ which is in fact implied by (1) and (2) in the cases of interest (see Remark 3 and Theorem 4). Among the uniqueness results mentioned above, only Kamin in [13] considers the case of a measure as initial data in the particular case of dimension 1 with a Dirac mass.

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Here our proof is quite general and not particular to R^N. It would carry over to equation (2) in a bounded domain Ω of R^N with Dirichlet or Neumann boundary conditions.

Section 1 is devoted to the proof of Theorem 1. In Section 2, we state an existence theorem for solutions of (P) whose initial value is a given nonnegative finite measure. We also study the dependence on the initial data.

Some last comments about motivations. Equation (2) arises in many applications. We will not recall them here since they can be found in most of the papers mentioned above or in the references they contain (see also [18] for a survey about the porous media equation). The case when the initial datum is a measure is also a model for physical phenomena (see [13] and [23, p. 677]). Moreover it arises in several mathematical questions. One example is the study of the asymptotic behavior for the solutions of the porous media equation which can be reduced to the uniqueness problem with a Dirac mass as initial data (see [11] and [15]).

SECTION 1

We denote by $C_c(R^N)$ (resp. $C_b(R^N)$) the set of continuous functions on R^N with compact support (resp. bounded) and by $\mathcal{M}(R^N)$ (resp. $\mathcal{M}^+(R^N)$) the set of finite (resp. and nonnegative) Radon measures on R^N. A sequence $\mu_n \subset \mathcal{M}(R^N)$ is said to be converging to μ in $\sigma(\mathcal{M}(R^N), C_c(R^N))$ (resp. $\sigma(\mathcal{M}(R^N), C_b(R^N))$) if, for any $\varphi \in C_c(R^N)$ (resp. $\varphi \in C_b(R^N)$)

$$\lim_{n\to\infty} \int_{R^N} \varphi \, d\mu_n = \int_{R^N} \varphi \, d\mu.$$

Case $N \geqslant 3$. Let $E_N(x) = 1/(N-2)S_N|x|^{N-2}$ where S_N is the surface of the unit N-sphere. For $u, \hat{u}$ solutions of (P) we denote

$$\text{a.e.} \quad t \in (0, T), \quad v(t) = E_N * u(t), \quad \hat{v}(t) = E_N * \hat{u}(t).$$

Since $u(t) \in L^1(R^N) \cap L^\infty(R^N)$, $v(t) \in C_b(R^N) \cap L^p(R^N)$ for $p > N/(N-2)$. (This follows from elementary properties of the convolution in R^N.)

LEMMA 2. If u is a nonnegative solution of (P), $u(t)$ converges in $\sigma(\mathcal{M}(R^N), C_b(R^N))$ to some $\mu \in \mathcal{M}^+(R^N)$ when $t \to 0$ essentially. Moreover when t decreases to 0, $v(t, x)$ increases to $v(0, x) = (E_N * \mu)(x)$ for all $x \in R^N$.

(Note that $v(0, x)$ is lower semi-continuous and is the usual potential of the finite measure μ.)

Proof of Lemma 2. The relation (2) implies that

$$\text{a.e.} \quad 0 < s \leqslant t \leqslant T, \quad u(t) - u(s) = \Delta \int_s^t \varphi(u(\sigma)) \, d\sigma \quad \text{in } \mathcal{D}'(R^N). \tag{4}$$

This is easily obtained by multiplying (2) by test-functions $\alpha_n(t)\,\theta(x)$, $\theta \in \mathcal{D}(R^N)$ and $\alpha_n(t) \in \mathcal{D}(0, T)$ converging to $1_{[s,t]}$ in a suitable way. Note that the assumptions on φ together with (1) imply

$$\varphi(u) \in L^1((\tau, T) \times R^N) \cap L^\infty((\tau, T) \times R^N), \quad \forall \tau \in (0, T). \tag{5}$$

Actually the relation (4) defines $u(t)$ for all $t \in (0, T]$. Moreover since $u(t) - u(s)$ and $\int_s^t \varphi(u(\sigma))\, d\sigma \in L^1(R^N)$, for all $0 < s \leqslant t \leqslant T$:

$$\int_{R^N} u(t) = \int_{R^N} u(s), \tag{6}$$

$$v(t) - v(s) = E_N * \left(\Delta \int_s^t \varphi(u(\sigma))\,d\sigma\right) = -\int_s^t \varphi(u(\sigma))\,d\sigma \leqslant 0 \quad \text{a.e. on } R^N. \tag{7}$$

In (7) we use a uniqueness result (see, e.g., [4, Lemma A.5]).

Relation (6) implies the relative compactness of $\{u(t); t \in (0, T)\}$ in $\sigma(\mathcal{M}(R^N), C_c(R^N))$. The monotonicity proves the uniqueness of the limit μ of $u(t)$ and the second part of the lemma (see, e.g., [17, Theorems 0.6, 3.8 and 3.9 about potentials of measures]). Moreover

$$v(t) \leqslant v(0) \Rightarrow \int_{R^N} -\Delta v(t) \leqslant \int_{R^N} -\Delta v(0).$$

Hence $\int_{R^N} u(t)$ converges to $\int_{R^N} -\Delta v(0) = \int_{R^N} d\mu$ and $u(t)$ converges to μ in $\sigma(\mathcal{M}(R^N), C_b(R^N))$.

Proof Theorem 1 *for* $N \geqslant 3$. Let $h \in]0, T[$ be fixed. If u and $\hat{u}$ are solutions of (P), by (4) we have

$\forall 0 < s \leqslant t < t + h < T,$

$$(u(t) - \hat{u}(t+h)) - (u(s) - \hat{u}(s+h)) = \Delta \int_s^t [\varphi(u(\sigma)) - \varphi(\hat{u}(\sigma + h))]\,d\sigma. \tag{8}$$

Letting s go to 0 gives in $\mathcal{D}'(R^N)$:

$$(u(t) - \hat{u}(t+h)) - (\mu - \hat{u}(h)) = \Delta \int_0^t [\varphi(u(\sigma)) - \varphi(\hat{u}(\sigma + h))]\,d\sigma. \tag{9}$$

Remark that $s \mapsto \int_s^t \varphi(u(\sigma))\,d\sigma$ is nondecreasing and $\Delta \int_s^t \varphi(u(\sigma))\,d\sigma$ is bounded in $L^1(R^N)$. By the results in [4] (Lemma A.5), it converges in $L^1_{\text{loc}}(R^N)$ to $\int_0^t \varphi(u(\sigma))\,d\sigma$. Let us denote

$$g(t) = \int_0^t [\varphi(u(\sigma)) - \varphi(\hat{u}(\sigma + h))]\,d\sigma + \hat{v}(h) - v(0).$$

Then (9) can be written as

$$u(t) - \hat{u}(t+h) = \Delta g(t) \quad (\Leftrightarrow g(t) = \hat{v}(t+h) - v(t)),$$

where $v(t)$ is defined in Lemma 2. This implies

$$g_t(t) = a(t)\,\Delta g(t), \tag{10}$$

where

$$a(t,x) = \begin{cases} \dfrac{\varphi(u(t,x)) - \varphi(\hat{u}(t+h,x))}{u(t,x) - \hat{u}(t+h,x)} & \text{if } u(t,x) \neq \hat{u}(t+h,x) \\ 0 & \text{if } u(t,x) = \hat{u}(t+h,x). \end{cases}$$

The function a is nonnegative and is in $L^\infty((\tau, T) \times R^N)$ for any $\tau \in (0, T)$. Hence g is solution of a *linear* equation; moreover if $\lim_{t \downarrow 0} u(t) = \lim_{t \downarrow 0} \hat{u}(t)$, $g(0) = \hat{v}(h) - v(0)$ is nonpositive by Lemma 2. If $a(.)$ were regular enough, by the maximum principle applied to (10) we would obtain

$$\forall 0 < t < t + h < T, \quad g(t) = \hat{v}(t+h) - v(t) \leqslant 0. \tag{11}$$

And that would imply $\hat{v} \leqslant v$, and, by a symmetric argument $\hat{v} = v$ and $\hat{u} = u$. What follows is to justify this maximum principle for equation (10) in our particular case. The method consists in multiplying (10) by the solution ψ of the dual problem $\psi_t + \Delta(a\psi) = 0$, $\psi(\bar{T}) = \theta \in \mathcal{D}^+(R^N)$, $0 < \bar{T} + h < T$, which formally gives

$$\int_{R^N} g(\bar{T})\theta = \int_{R^N} g(s)\psi(s).$$

($\mathscr{D}^+(R^N)$ denotes the space of nonnegative C^∞-functions with compact support in R^N).

Then, we show that the right-hand side has a nonpositive limit when $s \to 0$. The first step is to "solve" the above equation. For this let us approximate a by $a_p \in C^\infty([0, \bar{T}] \times R^N)$, nonnegative and satisfying:

$$a_p,\ |\nabla a_p|,\ \Delta a_p \text{ are bounded on } [0, \bar{T}] \times R^N \text{ for any } p,$$
$$\forall \tau \in (0, \bar{T}),\ a_p \text{ is bounded on } [\tau, \bar{T}) \times R^N \text{ independently of } p,$$
$$a_p \text{ converges to } a \text{ a.e.}(t, x) \in (0, \bar{T}) \times R^N.$$

(For instance, one can mollify a and multiply by a C^∞-function equal to 1 for $|x| \leqslant p$ and equal to 0 for $|x| \geqslant p + 1$.)

Then, $\varepsilon > 0$ being fixed we consider the dual problem

$$\frac{\partial}{\partial t}\psi_p + \Delta((a_p + \varepsilon)\psi_p) = 0, \qquad \psi_p(\bar{T}) = \theta \in \mathscr{D}^+(R^N), \tag{12}$$

where $0 < \bar{T} + h < T$. For simplicity we still denote $\bar{T}$ by T. It is well-known that this problem has a nonnegative C^∞-solution such that $\psi_p(t) \in L^1(R^N) \cap L^\infty(R^N)$ for all t (see [16]).

Now, multiply equation (10) by ψ_p and integrate to obtain:

$$\left.\begin{aligned} \int_{R^N} g(T)\theta - \int_{R^N} g(s)\,\psi_p(s) &= \int_s^T \int_{R^N} \left(g(\sigma)\frac{\partial\psi_p}{\partial t}(\sigma) + a(\sigma)\,\psi_p(\sigma)\,\Delta g(\sigma)\right) \mathrm{d}\sigma \\ &= \int_s^T \int_{R^N} (a - a_p - \varepsilon)\,\psi_p(\sigma)\,\Delta \mathrm{g}(\sigma)\,\mathrm{d}\sigma. \end{aligned}\right\} \tag{13}$$

In order to pass to the limit in p for $s \in (0, T)$, let us make some estimates on ψ_p. For convenience we denote

$$H_p(t) = E_N * \psi_p(t) \quad (\Rightarrow -\Delta H_p(t) = \psi_p(t)).$$

Multiplying (12) by $H_p(t)$ gives

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\frac{1}{2}\int_{R^N} |\nabla H_p(t)|^2 = \int_{R^N} (a_p + \varepsilon)\psi_p^2, \\ &\frac{1}{2}\int_{R^N} |\nabla\theta|^2 = \frac{1}{2}\int_{R^N} |\nabla H_p(t)|^2 + \int_s^T \int_{R^N} (a_p + \varepsilon)\psi_p^2. \end{aligned}\right\} \tag{14}$$

This proves that ψ_p is bounded in $L^2(0, T; R^N)$; it has a subsequence weakly converging to ψ_ε. Then $(a_p + \varepsilon)\psi_p$ also weakly converges to $(a + \varepsilon)\psi_\varepsilon$ in $L^2((\tau, T) \times R^N)$ for any $\tau \in (0, T)$. Hence the limit ψ_ε satisfies, in $\mathscr{D}'(R^N)$, an integrated form of (12), namely

$$\forall 0 < s < t \leqslant T, \quad \psi_\varepsilon(t) - \psi_\varepsilon(s) = -\Delta \int_s^t (a + \varepsilon)\psi_\varepsilon. \tag{15}$$

Since $\int_{R^N} \psi_p(t) = \int_{R^N} \theta$, $\psi_p(t)$ is bounded in $L^1(R^N)$. Hence ψ_ε is in $L^1(R^N)$ and by (15)

$$\int_{R^N} \psi_\varepsilon(t) = \int_{R^N} \theta. \tag{16}$$

Moreover, one can assume that $\psi_p(t)$ converges to $\psi_\varepsilon(t)$ in $\sigma(\mathscr{M}(R^N), C_b(R^N))$ for all $t > 0$.

Now (13) becomes:

$$\int_{R^N} g(T)\theta - \int_{R^N} g(s)\,\psi_\varepsilon(s) = -\varepsilon \int_s^T \int_{R^N} \psi_\varepsilon(\sigma)\,\Delta g(\sigma)\,\mathrm{d}\sigma. \tag{17}$$

(Remember that $\Delta g(t) = u(t) - \hat{u}(t+h)$ is bounded in any $L^p((s,T) \times R^N)$ and a_p converges to a a.e. on $(s,T) \times R^N$ and is uniformly bounded.)

Now we let $\varepsilon \to 0$. For any $s \in (0, T)$, the right-hand side of (17) is bounded by

$$\varepsilon \|\Delta g\|_{L^\infty((s,t)\times R^N)} \cdot \int_s^T \|\psi_\varepsilon(\sigma)\|_{L^1(R^N)},$$

which converges to 0 since ψ_ε is bounded in $L^\infty(0, T, L^1(R^N))$ by (16). If $H_\varepsilon(t) = E_N * \psi_\varepsilon(t)$, integrating (15) gives

$$H_\varepsilon(t) - H_\varepsilon(s) = \int_s^t (a + \varepsilon)\psi_\varepsilon.$$

By the nonnegativity of ψ_ε and H_ε this implies

$$0 \leqslant H_\varepsilon(t) \leqslant H_\varepsilon(T) = E_N * \theta.$$

Hence H_ε is bounded in $L^p((0,T) \times R^N)$ for $p > N/(N-2)$ and one can find convex combinations of these H_ε converging a.e. and strongly to H in $L^p((0,T) \times R^N)$ for some $p \in (N/(N-2), \infty)$. Since $\psi_\varepsilon(s)$ is uniformly bounded in $L^1(R^N)$, we can assume that the same combinations of $\psi(s)$ converge a.e. s in $\sigma(\mathscr{M}(R^N), C_c(R^N))$ to $v(s) \in \mathscr{M}^+(R^N)$. In order to pass to the limit in (17), we need a convergence in $\sigma(\mathscr{M}(R^N), C_b(R^N))$. This comes from the fact that

$$\forall s \in (0,T), \int_{R^N} \mathrm{d}v(s) = \int_{R^N} \theta = \lim_{\varepsilon\to 0} \int_{R^N} \psi_\varepsilon(s). \tag{18}$$

Indeed, for any $s \in (0,T)$, $\int_s^T (a+\varepsilon)\psi_\varepsilon$ is bounded in $L^1(R^N)$. Hence, by (15) there exists $\rho(s) \in \mathscr{M}^+(R^N)$ such that

$$\theta - v(s) = -\Delta\rho(s) \quad \text{in } \mathscr{D}'(R^N).$$

This together with (16) implies (18).

We finally obtain the existence of a family of nonnegative finite measures $\{v(s), s \in (0,T)\}$ such that

$$\forall s \in (0,T) \int_{R^N} g(T)\theta = \int_{R^N} g(s)\,\mathrm{d}v(s) \tag{19}$$

and

$$s \mapsto H(s) = E_N * v(s) \text{ is nondecreasing on } (0,T). \tag{20}$$

((20) comes from the monotonicity of H_ε.)

Now, $g(s) = \hat{v}(s+h) - v(s)$; by the monotonicity results of Lemma 2, (19) gives

$$\left.\begin{aligned}&\forall 0 < s < s_0,\\ &\int_{R^N} g(T)\theta \leqslant \int_{R^N} (\hat{v}(h) - v(s_0))\,\mathrm{d}\nu(s) = \int_{R^N} (\hat{u}(h) - u(s_0))H(s).\end{aligned}\right\} \tag{21}$$

(See the remarks below for the integration by parts.) But $\int_{R^N} \mathrm{d}\nu(s)$ is bounded independently of s and $H(s)$ decreases pointwise when s decreases to 0. Hence $\nu = -\Delta H(0^+) \in \mathscr{M}^+(R^N)$ and by (21)

$$\left.\begin{aligned}&\forall 0 < s_0,\\ &\int_{R^N} g(T)\theta \leqslant \int_{R^N} (\hat{u}(h) - u(s_0))H(0^+) = \int_{R^N} (\hat{v}(h) - v(s_0))\,\mathrm{d}\nu.\end{aligned}\right\} \tag{22}$$

Now letting s_0 decrease to 0 gives by monotonicity

$$\int_{R^N} g(T)\theta \leqslant \int_{R^N} (\hat{v}(h) - v(0))\,\mathrm{d}\nu. \tag{23}$$

But $v(0, x) = \hat{v}(0, x) \geqslant \hat{v}(h, x)$ for all $x \in R^N$. Hence

$$\forall \theta \in \mathscr{D}^+(R^N), \quad \int_{R^N} g(T)\theta \leqslant 0.$$

This implies the relation (11) we were looking for.

Remark 1. In the above we often use the fact that given $\mu, \nu \in \mathscr{M}^+(R^N)$:

$$\int_{R^N} (E_N * \mu)\,\mathrm{d}\nu = \int_{R^N} (E_N * \nu)\,\mathrm{d}\mu, \tag{24}$$

whatever this integral is finite or not. In (22), $H(0^+)$ is the decreasing limit of the potentials $H(s)$. It is generally not a l.s.c. potential itself but is equal *a.e.* to $E_N * \nu$. Since $\hat{u}(h) - u(s_0)$ is a "good" function, the integration by part works. It would not for $h = 0$, for $\hat{u}(0)$ is only a measure (see, e.g., [17] for more details).

Remark 2. The same method would give a similar uniqueness result for the equation

$$\begin{cases} u_t = \Delta\varphi(u) & \text{in } \mathscr{D}'((0, T) \times \Omega),\\ \text{‘}\varphi(u) = 0 & \text{on } \partial\Omega\text{’}\\ u(t) \to \mu & \text{in } \sigma(\mathscr{M}(R^N), C_b(R^N)),\end{cases}$$

with Ω a regular bounded open subset of R^N, by using the 'potentials' $v(t)$ solutions of

$$\begin{cases} -\Delta v(t) = u(t),\\ v(t) = 0 \quad \text{on } \partial\Omega\end{cases}$$

This method would clearly contain the cases $N = 1, 2$.

Remark 3. There is no potential in R^N if $N = 1.2$. Hence we have to do the above computations in an 'approximated' way using the solutions of

$$\alpha v_\alpha(t) - \Delta v_\alpha(t) = u(t),$$

and letting α go to 0. This requires more *a priori* assumptions on the solution u, namely $\varphi(u) \in L^1((0, T) \times R^N)$. This is implied by (1) and (2) in most cases of interest, like the porous media case $\varphi(r) = r^m$ (see Theorem 4) or the Stefan problem case $\varphi(r) = (r-1)^+$ (since $u \in L^1((0, T) \times R^N)$). It was also proved in [14], that this always holds in dimension 1 with very weak assumptions on φ.

Proof of Theorem 1 for $N = 1, 2$. Since there is no potential in R^N if $N = 1, 2$, we will use the solutions of

$$\forall t > 0, \quad \alpha v_\alpha(t) - \Delta v_\alpha(t) = u(t), \quad \alpha \hat{v}_\alpha(t) - \Delta \hat{v}_\alpha(t) = \hat{u}(t). \tag{25}$$

For $\alpha > 0$, $(\alpha - \Delta)^{-1}$ is a 'good' operator even when $N = 1, 2$ and in particular (see, e.g., [4, Lemma 1.1]):

$$u(t) \in L^1(R^N) \cap L^\infty(R^N) \Rightarrow v_\alpha(t) \in L^1(R^N) \cap C_b(R^N).$$

We will denote K_α the kernel associated with $(\alpha - \Delta)^{-1}$, i.e., $v_\alpha(t) = K_\alpha * u(t)$. The result corresponding to Lemma 2 is:

LEMMA 3. If $u \geq 0$ is a solution of (P), $u(t)$ converges in $\sigma(\mathcal{M}(R^N), C_b(R^N))$ to some $\mu \in \mathcal{M}^+(R^N)$ when $t \to 0$. Moreover when t decreases to 0, $v_\alpha(t, x) + \alpha K_\alpha * \int_t^\tau \varphi(u(\sigma))\,d\sigma$ increases to $K_\alpha * (\mu + \alpha \int_0^\tau \varphi(u(\sigma))\,d\sigma(x))$ for all $x \in R^N$ and all $\tau \in (0, T)$.

Note that for $\tau \in (0, T)$ fixed, $K_\alpha * \int_t^\tau \varphi(u(\sigma))\,d\sigma$ is continuous and when t decreases to 0, it increases to the l.s.c. function $K_\alpha * \int_0^\tau \varphi(u(\sigma))\,d\sigma$ which is well-defined since $\int_0^\tau \varphi(u(\sigma))\,d\sigma \in L^1(R^N)$ by assumption on φ.

If $w_\alpha(t) = v_\alpha(t) + \alpha K_\alpha * \int_t^\tau \varphi(u(\sigma))\,d\sigma$, we have for $0 < s < t$:

$$\begin{aligned} w_\alpha(t) - w_\alpha(s) &= K_\alpha * \left(u(t) - u(s) - \alpha \int_s^t \varphi(u(\sigma))\,d\sigma\right) \\ &= K_\alpha * \left((\Delta - \alpha) \int_s^t \varphi(u(\sigma))\,d\sigma\right) = -\int_s^t \varphi(u(\sigma))\,d\sigma \leq 0 \quad \text{a.e.} \end{aligned} \tag{26}$$

This proves the second part of the lemma. Then we finish as in Lemma 2 using that

$$\int_{R^N} u(t) = \int_{R^N} \alpha v_\alpha(t) \to \int_{R^N} \alpha v_\alpha(0) = \int_{R^N} d\mu.$$

Now to replace the function g of the previous proof, we introduce for $h > 0$ fixed:

$$g_\alpha(t) = \int_0^t G(\sigma)\,d\sigma + \hat{v}_\alpha(h) - v_\alpha(0) - \alpha(\alpha - \Delta)^{-1} \int_0^t G(\sigma)\,d\sigma,$$

where we denote $G(\sigma) = \varphi(u(\sigma)) - \varphi(\hat{u}(\sigma + h))$. Then we verify

$$(\alpha - \Delta) g_\alpha(t) = \hat{u}(t + h) - u(t) \quad (\Leftrightarrow g_\alpha(t) = \hat{v}_\alpha(t + h) - v_\alpha(t)),$$

$$g_{\alpha t} + a(\alpha - \Delta) g_\alpha(t) = -\alpha(\alpha - \Delta)^{-1} G(t) \leq \alpha K_\alpha * \varphi(\hat{u}(t + h)), \tag{27}$$

where a is defined in the previous proof.

Now, we do the same computations as for $N \geqslant 3$ with the operator $\alpha - \Delta$ instead of $-\Delta$ and we obtain—like in (19) and (20)—the existence of $\{v_\alpha(s) \in \mathscr{M}^+(R^N); s \in (0, T)\}$ such that, for $\theta \in \mathscr{D}^+(R^N)$:

$$\left.\begin{array}{l}\forall s \in (0, T),\\ \displaystyle\int_{R^N} g_\alpha(T)\theta \leqslant \int_{R^N} g_\alpha(s)\,\mathrm{d}v_\alpha(s) + \int_s^T \int_{R^N} \alpha K_\alpha * \varphi(\hat{u}(\sigma + h))\,\mathrm{d}v_\alpha(\sigma),\end{array}\right\} \tag{28}$$

$$s \longmapsto H_\alpha(s) = K_\alpha * v_\alpha(s) \quad \text{is nondecreasing on } (0, T). \tag{29}$$

By Lemma 3, for all $\tau \in (0, T)$ and $0 < s < s_0$:

$$g_\alpha(s) \leqslant \left[\hat{v}_\alpha(h) + \alpha K_\alpha * \int_0^\tau \varphi(\hat{u}(\sigma + h))\,\mathrm{d}\sigma\right] - \left[v_\alpha(s_0) + \alpha K_\alpha * \int_{s_0}^\tau \varphi(u(\sigma))\,\mathrm{d}\sigma\right] + \alpha K_\alpha * \int_s^\tau G(\sigma)\,\mathrm{d}\sigma. \tag{30}$$

Now we can pass to the limit when $s \to 0$ as in (22). The last term of (30) is easily controlled after integration by part.

$$\begin{aligned}\varlimsup_{s \downarrow 0} \int_{R^N} g_\alpha(s)\,\mathrm{d}v_\alpha(s) &\leqslant \int_{R^N}\left[\hat{v}_\alpha(h) + \alpha K_\alpha * \int_0^\tau \varphi(\hat{u}(\sigma + h))\,\mathrm{d}\sigma - v_\alpha(s_0) - \alpha K_\alpha * \int_{s_0}^\tau \varphi(u(\sigma))\,\mathrm{d}\sigma\right]\mathrm{d}v_\alpha\\ &\quad + \int_{R^N} \alpha H_\alpha(0^+) \int_0^\tau G(\sigma)\,\mathrm{d}\sigma,\end{aligned}$$

with $v_\alpha = \lim_{s \downarrow 0} v_\alpha(s)$ and $H_\alpha(0^+) = K_\alpha * v_\alpha$ a.e. We let s_0 decrease to 0 above, use the monotonicity established in Lemma 3 to obtain

$$\varlimsup_{s \downarrow 0} \int_{R^N} g_\alpha(s)\,\mathrm{d}v_\alpha(s) \leqslant \int_{R^N} \alpha H_\alpha(0^+) \int_0^h \varphi(\hat{u}(\sigma))\,\mathrm{d}\sigma.$$

Finally, coming back to (28) and remarking that $H_\alpha(s) \leqslant H_\alpha(T) = K_\alpha * \theta$ we have, for any $\theta \in \mathscr{D}^+(R^N)$,

$$\int_{R^N} g_\alpha(T)\theta \leqslant \int_{R^N} \alpha\theta K_\alpha * \int_0^h \varphi(\hat{u}(\sigma))\,\mathrm{d}\sigma + \int_0^T \int_{R^N} \alpha\theta K_\alpha * \varphi(\hat{u}(\sigma + h))\mathrm{d}\sigma.$$

Hence

$$g_\alpha(T) \leqslant \alpha K_\alpha * \int_0^{T+h} \varphi(\hat{u}(\sigma))\,\mathrm{d}\sigma.$$

Now we let h, then α go to 0. For any $f \in L^1(R^N)$, $\alpha w_\alpha = \alpha K_\alpha * f$ converges to 0 in $\mathscr{D}'(R^N)$ when $\alpha \to 0$. Indeed, $\int_{R^N} \alpha w_\alpha = \int_{R^N} f$ implies the convergence of $\alpha_n w_{\alpha n}$ to some $v \in \mathscr{M}^+(R^N)$. The relation $\alpha^2 w_\alpha - \Delta(\alpha w_\alpha) = \alpha f$ implies that $\Delta v = 0$. Hence $v = 0$. Coming back to the definition of $g_\alpha(T)$, we obtain

$$\int_0^T \varphi(u(\sigma)) - \varphi(\hat{u}(\sigma)) \leqslant 0.$$

This completes the proof.

Remark. Thanks to Lemmas 2 and 3, the condition (3) in Theorem 1 can be weakened to the requirement

$$\lim_{h \downarrow 0} \frac{1}{h} \int_0^h (u(t) - \hat{u}(t))\, dt = 0 \quad \text{in } \mathscr{D}'(R^N). \tag{3$'$}$$

This may be useful in view of the existing literature where the solutions are often defined as functions u satisfying (1) and

$$\int_0^T \int_{R^N} u\psi_t + \varphi(u)\,\Delta\psi + \int_{R^N} u(0)\,\psi(0) = 0$$

for any $\psi \in C_0^\infty([0, T[\times R^N)$. Clearly two solutions u and $\hat{u}$ of the above satisfy (2) and (3)$'$.

SECTION 2

Existence results and dependence on the initial data.

THEOREM 4. Let $m \geqslant 1$ and $\mu \in \mathscr{M}^+(R^N)$. Then there exists a unique nonnegative $u \in C((0, \infty); L^1(R^N)) \cap L^\infty((\tau, \infty) \times R^N)$ for all $\tau > 0$ such that

$$u_t = \Delta u^m \quad \text{in } \mathscr{D}'((0, \infty) \times R^N) \tag{31}$$

$$u(t) \to \mu \text{ in } \sigma(\mathscr{M}(R^N), C_b(R^N)) \quad \text{when } t \downarrow 0. \tag{32}$$

If $\hat{u}$ is another such solution with initial data $\mu \in \mathscr{M}^+(R^N)$

$$\forall t \in (0, \infty) \int_{R^N} |u(t) - \hat{u}(t)| \leqslant \int_{R^N} |\mu - \hat{\mu}|. \tag{33}$$

Moreover, if $\mu_n \in \mathscr{M}^+(R^N)$ converges to μ in $\sigma(\mathscr{M}(R^N), C_b(R^N))$, the associated solutions $u_n(t)$ converge to $u(t)$ in $L^1(R^N)$ for all $t > 0$.

Remark 4. If μ is the Dirac mass δ at the origin, the solution of (31), (32) has been explicitly determined (see Barenblatt [2]). It is given by

$$u(t, x) = t^{-k}\left[\left(a - \frac{k(m-1)x^2}{2mNt^{k/N}}\right)^+\right]^{1/(m-1)}$$

where $k^{-1} = m - 1 + (2/N)$ and a is a constant depending on m and N in such a way that $\int_{R^N} u(t) = 1$.

Remark 5. The proof of the above result contains several ingredients. First the existence of a solution to (31) when the initial data is regular. This was proved in [20]. It can also be obtained as a consequence of the abstract theory of evolution equations governed by accretive operators which carries over to the more general equation (2) (see [5]). Then, approximating μ by 'regular' functions μ_n, one has to prove that the solution of (31) with initial data μ_n converges to the solution of (31)–(32). This needs some compactness arguments that can be obtained through two different ways. One can use that

$$\int_{R^N} |\Delta u^m| = \int_{R^N} |u_t| \leqslant \frac{C}{t} \int_{R^N} u_0$$

as proved in [1]. Actually this method would apply to (2) for the general class of functions φ described in [8]. Here we will use, in conjunction with some direct estimates in (31), the L^∞-regularizing effect which says that the solution of (31) belongs to $L^\infty((\tau, \infty) \times R^N)$ for any $\tau > 0$. The latter property—which is needed to apply our uniqueness result—is also true for equation (2) with very weak assumptions on φ. To illustrate the generality of this method let us establish, at least for $N \geqslant 3$, a more general existence result.

Let us consider $\varphi : [0, \infty) \to [0, \infty)$ locally Lipschitz, nondecreasing, $\varphi(0) = 0$. Assume $N \geqslant 3$ and for instance (see [3]):

$$\exists \alpha > \frac{N-2}{N} \quad \text{such that } (\varphi(r))^{1/\alpha} \text{ is convex for } r \text{ large.} \tag{34}$$

Then we have:

PROPOSITION 5. For all $\mu \in \mathcal{M}^+(R^N)$, there exists a unique nonnegative $u \in L^\infty((0, \infty); L^1(R^N)) \cap L^\infty((\tau, \infty) \times R^N)$ for all $\tau > 0$, solution of

$$u_t = \Delta\varphi(u) \quad \text{in } \mathcal{D}'((0, \infty) \times R^N)$$
$$u(t) \to \mu \quad \text{in } \sigma(\mathcal{M}(R^N), C_b(R^N)) \quad \text{when } t \downarrow 0.$$

Moreover the estimate (33) holds.

Remark 6. Assumption (34) insures that $u \in L^\infty((\tau, \infty) \times R^N)$ for $\tau > 0$. A slightly different assumption can be found in [21].

Proof of Proposition 5. Let $\mu_n \in L^1(R^N)$ nonnegative and converging to μ in $\sigma(\mathcal{M}(R^N), C_b(R^N))$. By the existence results in [5] and the L^∞-regularizing effects established in [3], there exists $u_n \in C([0, \infty); L^1(R^N)) \cap L^\infty((\tau, \infty) \times R^N)$ for any $\tau > 0$ such that

$$u_{nt} = \Delta\varphi(u_n) \quad \text{in } \mathcal{D}'((0, \infty) \times R^N) \tag{35}$$
$$u_n(0) = \mu_n,$$
$$\|u_n(t)\|_{L^\infty} \leqslant K + \left(\frac{K_0}{t}\right)^\gamma, \quad K, K_0, \gamma > 0, \tag{36}$$

where K, K_0, γ depend only on $\|\mu_n\|_{L^1}$, α, φ and N. Remark that u_n is uniformly bounded in $C([0, \infty); L^1(R^N))$ since $\int_{R^N} u_n(t) = \int_{R^N} \mu_n$ by (35). Let us make some formal estimates now. Multiplying (35) by $\varphi(u_n)_t$ yields:

$$\tfrac{1}{2}\frac{\partial}{\partial t}\int_{R^N} |\nabla\varphi(u_n)|^2 = -\int_{R^N} u_{nt}\varphi(u_n)_t \leqslant 0. \tag{37}$$

Multiplying (35) by $\varphi(u_n)$ and integrating give for $0 < \tau < t$

$$\int_\tau^t \int_{R^N} |\nabla\varphi(u_n)|^2 = \int_{R^N} \psi(u_n(\tau)) - \psi(u_n(t)), \tag{38}$$

where $\psi(r) = \int_0^r \varphi(\sigma)\,d\sigma$. Since $u_n(\tau)$ is bounded in L^∞ independently of n (see (36)), this implies that $\nabla\varphi(u_n)$ is uniformly bounded in $L^2{}_{\text{loc}}((0, \infty) \times R^N)$. By (37), $\nabla\varphi(u_n)$ is even uniformly bounded in $L^\infty(\tau, \infty; L^2(R^N))$ for any $\tau > 0$. Hence, integrating (37) proves that $\int_\tau^\infty \int_{R^N} u_{nt}\varphi(u_n)_t$ is bounded for any $\tau > 0$. Since φ is locally Lipschitz, we obtain that $\int_\tau^\infty \int_{R^N} [\varphi(u_n)_t]^2$ is bounded. Finally we deduce that

$$\left.\begin{array}{l}\varphi(u_n) \text{ is bounded in } W^{1,2}_{\text{loc}}((0, \infty) \times R^N)\\ \text{the bound depends only on } \sup\limits_n \|\mu_n\|_1.\end{array}\right\} \tag{39}$$

The formal computation (38) can be justified like in [5, Proposition 10]. For the other ones, we use $\varphi(u(t + h)) - \varphi(u(t))$ and let h go to 0.

By (39), there exists a subsequence (still denoted $\varphi(u_n)$) converging in $L^2{}_{\text{loc}}((0, \infty) \times R^N)$ and a.e. to some w. On the other hand, a subsequence of u_n converges weakly in $L^2(K)$ for any compact $K \subset (0, \infty) \times R^N$ and the limit u satisfies $w(t, x) = \varphi(u(t, x))$ a.e. since φ is a maximal monotone operator in $L^2(K)$ (see [7, Proposition 2.5]). Clearly

$$u \in L^\infty(0, T, L^1(R^N)) \cap L^\infty((\tau, \infty) \times R^N)$$

for all $\tau > 0$ and satisfies

$$u_t = \Delta\varphi(u) \quad \text{in } \mathscr{D}'((0, \infty) \times R^N).$$

It remains to show that $\hat{\mu} = \lim\limits_{t\to 0} \operatorname{ess} u(t)$ (in $\sigma(\mathscr{M}(R^N), C_b(R^N))$) which exists by Lemma 1, is equal to μ (note that $\int d\hat{\mu} \leqslant \int d\mu$).

For this, let us assume we have chosen $\mu_n = \mu * \rho_n$ with $\rho_n(x) = \lambda_N n^N \rho(n|x|)$, $\forall x \in R^N$ where $\rho \in C^\infty([0, \infty)]$ is supported in $[0, 1]$ and λ_N is a constant such that $\int_{R^N} \rho_n = 1$. With this choice (see [17]) $n \to v_n^0 = E_N * \mu_n$ is nondecreasing (and v_n^0 increases pointwise to $v^0 = E_N * \mu$). Hence

$$n \to v_n(t) = E_N * u_n(t) \text{ is nondecreasing.}$$

Indeed $v_n(t)$ is a solution of $v_{nt} + \varphi(-\Delta v_n) = 0$, $v_n(0) = v_n^0$ and one can use the maximum principle for this equation (see, e.g., [9]). Since $\int_{R^N} -\Delta v_n(t) = \int_{R^N} u_n(t)$ is bounded, $v_n(t)$ increases to a potential $v(t)$ such that $-\Delta v(t)$ is the limit (in $\sigma(\mathscr{M}(R^N), C_c(R^N))$) of $-\Delta v_n(t)$ (see, e.g., [17]). Necessarily $v(t) = E_N * u(t)$ (at least a.e. t). Now, by Lemma 2, if $\hat{v}^0 = E_N * \hat{\mu}$, we have

$$\begin{aligned}\hat{v}^0 \geqslant v(t) \geqslant v_n(t), \quad \forall n, \quad \text{a.e. } t\\ \Rightarrow \hat{v}^0 \geqslant v_n^0, \quad \hat{v}^0 \geqslant v^0.\end{aligned}$$

On the other hand

$$\begin{aligned}v^0 \geqslant v_n^0 \geqslant v_n(t), \quad \forall n, \quad \text{a.e. } t\\ \Rightarrow v^0 \geqslant v(t), \quad v^0 \geqslant \hat{v}^0.\end{aligned}$$

Hence $v_0 = \hat{v}_0$ and $\mu = \hat{\mu}$.

To complete the proof, let us prove (33). By accretivity in $L^1(R^N)$ (see [5]), for any n, t:

$$\int_{R^N} |u_n(t) - \hat{u}_n(t)| \leqslant \int_{R^N} |\mu_n - \hat{\mu}_n| = \int_{R^N} |\rho_n * (\mu - \hat{\mu})| \leqslant \int_{R^N} |\mu - \hat{\mu}|. \tag{40}$$

We apply a Fatou-type lemma to finish.

Proof of Theorem 4. For $N \geqslant 3$, the existence of u is a consequence of the Proposition 3.5. Using the particular structure of $\varphi(u) = u^m$, we add an argument to the previous proof in order to absorb the cases $N = 1, 2$ and to prove the continuity results for all N.

Let μ_n and u_n defined as in this proof. The estimates we established are valid for any N. Hence (39) holds and a subsequence of u_n converges a.e. to $u \in L^\infty((\tau, \infty) \times R^N) \cap L^\infty(0, T, L^1(R^N))$ solution of (31). Moreover, in this particular case, we have

$$\lim_{s \downarrow 0} \int_0^T \int_{R^N} u_n^m(\sigma)\, \mathrm{d}\sigma = 0 \quad \text{uniformly in } n.$$

Indeed, by the L^∞-estimate (see [3, 21])

$$\|u_n(t)\|_{L^\infty} \leqslant \frac{C}{t^\sigma} \|\mu_n\|_{L^1}^\delta \quad \text{with } \sigma = \frac{1}{m - 1 + (2/N)} \quad \delta = \frac{2k}{N}.$$

Hence

$$\int_0^T \int_{R^N} u_n^m(\sigma)\, \mathrm{d}\sigma \leqslant C \|\mu_n\|_{L^1}^{1 + \delta(m-1)} \int_0^T \frac{\mathrm{d}t}{t^{(m-1)\sigma}} \quad \text{where } (m - 1)\sigma < 1. \tag{41}$$

In particular, for all t, $\int_0^t u^m(\sigma)\, \mathrm{d}\sigma \in L^1(R^N)$ and is the limit in $L^1_{\mathrm{loc}}(R^N)$ of $\int_0^t u_n^m(\sigma)\, \mathrm{d}\sigma$. By Lemmas 2 and 3, $u(t)$ converges in $\sigma(\mathcal{M}(R^N), C_b(R^N))$ to some $\hat{\mu}$ when $t \downarrow 0$. Now, we can pass to the limit in

$$u_n(t) - \mu_n = \Delta \int_0^t u_n^m(\sigma)\, \mathrm{d}\sigma \quad \text{in } \mathscr{D}'(R^N)$$

and obtain that $\mu = \hat{\mu}$.

Remark that $\int_{R^N} u_n(t) = \int_{R^N} \mu_n \to \int_{R^N} \mu = \int_{R^N} u(t)$. Hence $u_n(t)$ converges to $u(t)$ in $L^1(R^N)$ for a.e. t and even $\forall t > 0$ by the contraction property. The uniqueness proves the convergence of the whole sequence.

By (39) for any open subset Ω relatively compact in $(0, \infty) \times R^N$

$$\|u^m\|_{W^{1,2}(\Omega)} \leqslant C\left(\int_{R^N} \mu\right). \tag{42}$$

And (41) gives

$$\int_0^T \int_{R^N} u^m(\sigma)\, \mathrm{d}\sigma \leqslant C \cdot T^{1 - \sigma(m-1)} \cdot \left[\int_{R^N} \mu\right]^{1 + \delta(m-1)}. \tag{43}$$

For the uniqueness when $N = 1, 2$, given u, $\hat{u}$ solutions of (31) and (32) we apply Theorem 1 to $u(. + \tau)$ (as well as $\hat{u}(. + \tau)$) for any $\tau > 0$ to prove that they coincide with the solutions in the sense of semigroups. Hence (43) holds for $u, \hat{u}$ and we can apply Theorem 1 to u and $\hat{u}$.

Now, let μ_n converge to μ in $\sigma(\mathcal{M}(R^N), C_b(R^N))$ and u_n, u the corresponding solutions. The same arguments as above using the estimates (42) and (43) plus the uniqueness result prove that $u_n(t)$ converges to $u(t)$ in $L^1(R^N)$.

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